An introduction to C-minimal structures and their cell decomposition theorem

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Abstract

Developments in valuation theory, specially the study of algebraically closed valued fields, have used the model theory of C-minimal structures in different places (specially the work of Hrushovski-Kazdhan in [HK] and Haskell-Hrushovski-Macpherson in [HHM]). We intend with this text both to divulgate a basic comprehension of C-minimality for those mathematicians interested in valuation theory having a basic knowledge in model theory and to provide a slightly different presentation of the cell decomposition theorem proved by Haskell and Macpherson in [HM94].

Studying algebraic structures from a model-theoretic point of view can be described as studying the category of definable sets of algebraic structures: objects correspond to definable sets (i.e., solution sets of a particular first order formula) and morphisms correspond to definable functions (i.e., functions for which the graph is a definable set). A model theoretic perspective allows different ways of generalizing properties one can extract from algebraic structures. For instance, quantifier elimination for real closed fields $(R, \leq, +, \cdot, 0, 1)$ implies that definable subsets of R are exactly the semi-algebraic sets but also induced the fruitful notion of o-minimality: an ordered structure $(M, \leq, ...)$ is o-minimal if every definable subset of M is a finite union of points and intervals. In the same spirit, quantifier elimination for theories of valued structures like algebraically closed valued fields or the p-adic fields induce different notions of minimality, C-minimality being one of them. The aim of the text is to provide the reader with a basic comprehension of C-minimality (hopefully giving her tools to easy the reading of articles like [HK, HHM]), and to expose a proof of a deep theorem proved by Haskell and Macpherson in [HM94]: the cell decomposition theorem for dense C-minimal structures. We do not present new results and most of the article follows the same scheme as [HM94] though most of the proofs (and some definitions) have been simplified (and in some cases corrected). Section 1 contains a brief introduction to C-minimality together with definitions, examples and basic properties. In section 2 we define what cells are and study definable functions. Finally the cell decomposition theorem is proved in section 3 together with some results about dimension.

Notation will be standard with the following remarks. Capital L is restricted for first-order languages (with all possible subscripts and superscripts like L', L_0, etc.). For a set A, $L(A)$ is the expansion of L with a new constant for every element in A. We say a formula ϕ has parameters from A if it is an $L(A)$-formula. Given an L-structure M, $A \subseteq M^n$ and a formula $\phi(x)$ with $|x| = n$ (the length of the tuple), we denote by $\phi(A)$ the set $\{a \in A : M \models \phi(a)\}$. We say A is definable if there is an $L(M)$-formula $\phi(x)$

(i.e., allowing parameters from M) such that $A = \phi(M)$. If ϕ is an L-formula we also say A is 0-definable. We allow a handy ambiguity using M both for an L-structure and its universe. The automorphism group of M is denoted by $Aut(M)$. For an ultrametric space M with map $d : M^2 \to \Gamma \cup \{\infty\}$ we will denote $\Gamma \cup \{\infty\}$ by dM and assume the function d is always surjective. If the ultrametric comes from a valuation function v, we also denote dM by vM. The use of Γ will be restricted to other purposes through the text.

1 Introduction to C-minimality: C-sets, trees and C-minimal structures

We start this section introducing C-sets and good trees:

Definition 1. Let $C(x, y, z)$ be a ternary relation. A *C-set* is a structure (M, C) satisfying axioms (C1)-(C4):

(C1) $\forall xyz(C(x, y, z) \to C(x, z, y))$,

(C2) $\forall xyz(C(x, y, z) \to \neg C(y, x, z))$,

(C3) $\forall xyzw(C(x, y, z) \to (C(w, y, z) \vee C(x, w, z)))$,

(C4) $\forall xy(x \neq y \to C(x, y, y))$,

(D) $\forall xy(x \neq y \to \exists z(z \neq y \wedge C(x, y, z)))$.

If in addition (M, C) satisfies axiom (D) we say it is a *dense* C-set.

Examples 2.

- The trivial C-relation on a set M defined by $C(x, y, z) \Leftrightarrow x \neq y = z$.
- For every ultrametric $d : M^2 \to \Gamma \cup \{\infty\}$ there is a C-relation defined by $C(x, y, z) \Leftrightarrow d(x, y) < d(y, z)$. In particular, for a valued structure there is an associated C-relation defined by the induced ultrametric $d(x, y) := v(x - y)$, i.e., $C(x, y, z) \Leftrightarrow v(x - y) < v(y - z)$.
- For T a tree and A a set of branches of T (i.e., maximal chains of T), there is a C-relation on A defined by $C(x, y, z) \Leftrightarrow x \cap y = x \cap z \subset y \cap z$.

Let $(T, \leq, \inf, F)$ be a meet semi-lattice tree where $\inf(a, b)$ denotes the meet of a and b and F is a unary predicate denoting the leaves of T (we will abuse notation letting $\inf(A, b) := \sup\{\inf(a, b) : a \in A\}$ if existing).

Definition 3. A *good tree* is a structure $(T, \leq, \inf, F)$ satisfying axioms $(T1) - (T4)$:

(T1) $(T, \leq, \inf)$ is a meet semi-lattice tree,

(T2) $\forall x(F(x) \leftrightarrow \neg\exists y(x < y))$ (F is the set of leaves),

(T3) $\forall x \exists y(x \leq y \wedge F(y))$ (T has leaves everywhere),

(T4) $\forall x(\neg F(x) \to \exists yz(y \neq z \wedge x < y \wedge x < z))$ (every point which is not a leaf branches),

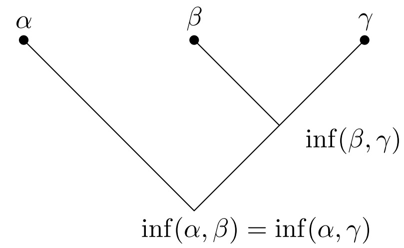
(D') $\forall xy(F(x) \wedge F(y) \to \exists w(w \neq y \wedge F(w) \wedge \inf(x, y) < \inf(y, z)))$ (the set of leaves is dense).

If in addition it satisfies (D') then it is called a *dense* good tree.

Theorem (Adeleke, Neumann, Delon)**.** *C-sets and good trees are bi-interpretable classes.*

Bi-interpretability, essentially means that each structure can be recovered as a quotient of the other by a definable equivalence relation. Adeleke and Neumann shown one direction of this theorem in [AN98] for dense C-sets and the statement as it is presented here is due to Delon in [Del11]. We briefly sketch the construction. For a good tree T, we define a C-relation on the set of leaves F by:

$$C(x, y, z) \Leftrightarrow \inf(x, y) = \inf(x, z) < \inf(y, z).$$



For the converse, if (M, C) is a C-set, there is a good tree denoted $T(M)$ and called *the canonical tree of* M, which is interpretable in M, having as its universe the set of equivalence classes of elements of M^2 modulo the equivalence relation $\sim$ defined by:

$$(a_1, a_2) \sim (b_1, b_2) \text{ iff } M \models \neg C(a_1, b_1, b_2) \wedge \neg C(a_2, b_1, b_2) \wedge \neg C(b_1, a_1, a_2) \wedge \neg C(b_2, a_1, a_2).$$

The set of leaves of $T(M)$ equipped with the C-relation above defined is definably isomorphic to M. This allows us to identify M in $T(M)$ with the set of leaves F and implies in particular that an embedding of C-sets $f : M \to N$ induces an embedding of good trees $\hat{f} : T(M) \to T(N)$ and that the automorphism groups $Aut(M)$ and $Aut(T(M))$ are canonically isomorphic. In all [AN98, HM94, MS96] C-sets were by assumption dense. Without the density assumption we still have that (M, C) is dense if and only if $T(M)$ is dense. In an ultrametric space M having a C-relation defined as in example 2, $T(M)$ corresponds to a tree where each branch is isomorphic to a copy of the ordered set dM. It is also isomorphic to the set of closed balls with inclusion as its order.

A C-structure is simply a C-set with possibly extra structure. In what follows we work in a fixed C-structure M. We use lower case Greek letters α, β, γ to denote both elements of M and leaves in $T(M)$ and lower case letters a, b, c to denote arbitrary elements in $T(M)$ (contrary to the usual use in valuation theory). From now on we let $T := T(M) \setminus F$, that is, all the elements in $T(M)$ which are not leaves. For $a \in T$, we define an equivalence relation E_a on $(a_>)$ (i.e. $\{b \in T(M) : b > a\}$) by

$$xE_ay \Leftrightarrow a < \inf(x, y).$$

Equivalences classes are called *cones at a*. *The branching number of a*, denoted by $bn(a)$, is the number of equivalence E_a-classes. For $a, b \in T$ such that $a < b$, *the cone of b at a*, denoted by $\Gamma_a(b)$, is the E_a-class of b. We abuse notation using $\Gamma_a(b)$ to denote also the subset of M defined by $\Gamma_a(b) \cap F$ (again identifying the set of leaves F with M). In particular, for $\alpha \in M$ and $a \in T(M)$ such that $a \leq \alpha$, the cone $\Gamma_a(\alpha)$ will be usually taken to be the set $\{\beta \in M : a < \inf(\alpha, \beta)\}$. For $\alpha, \beta \in M$, we then have that

$$\Gamma_{\inf(\alpha,\beta)}(\alpha) = \{x \in M : M \models C(\beta, x, \alpha)\}.$$

In an ultrametric space cones correspond to open balls. For example, if K is a valued field and we add a symbol for the C-relation defined as in the example 2, we have that

$$\Gamma_{\inf(\alpha,\beta)}(\alpha) = \{x \in K : K \models C(\beta, x, \alpha)\} = \{x \in K : K \models v(\alpha - \beta) < v(x - \alpha)\}.$$

For practical reasons we treat M as a cone at ∞, that is, we extend $T(M)$ by adding a new element $-\infty$ satisfying $-\infty < a$ for all $a \in T(M)$ and we let $\Gamma_{-\infty}(\alpha) := M$ for all $\alpha \in M$. For $a \in T(M)$ an *n-level set at a* corresponds to the set $\{x \in T : a \leq x\}$ with n cones at a removed. For $n > 0$ we require that $bn(a) > n$. In particular, for $a \in T(M)$, the 0-level set at *the 0-level set at a* is denoted in symbols by Λ_a. If $a \in T$, Λ_a corresponds to the union of all cones at a; if $a = \alpha \in M$, then $\Lambda_\alpha = \{\alpha\}$. For $\alpha, \beta \in M$ we have that

$$\Lambda_{\inf(\alpha,\beta)} = \{x \in M : M \models \neg C(x, \alpha, \beta)\}.$$

0-level sets correspond in ultrametric spaces to closed balls, for instance,

$$\Lambda_{\inf(\alpha,\beta)} = \{x \in M : M \models \neg C(x, \alpha, \beta)\} = \{x \in M : M \models d(x, \alpha) \geq d(\alpha, \beta)\}.$$

For $b_1, ..., b_n \in T$ such that $a < b_i$ and $\neg b_i E_a b_j$ for all $1 \leq i < j \leq n$, the expression $\Lambda_a(b_1, ..., b_n)$ denotes the n-level set where the n cones removed correspond to $\Gamma_a(b_i)$ for $1 \leq i \leq n$ (1-level sets are called "thin annulus" in [HK]). Both for cones an n-level sets, the point a is called its *basis* and we let $\min(\cdot)$ to be the function sending cones and n-level sets to their bases. Finally, for $a, b \in T(M) \cup \{-\infty\}$, such that $a < b$, *the interval* (a, b) denotes in $T(M)$ the set $\{x \in T : a < x < b\}$ and in M the set $\Gamma_a(b) \setminus \Lambda_b$. Cones, intervals and n-level sets can be seen as subsets of $T(M)$ or M and we usually let the context specify which one is intended. We denote the set of all cones (including M) by $\mathcal{C}$, the set of all intervals by $\mathcal{I}$ and the set of all n-levels by $\mathcal{L}_n$ for $n < \omega$. In M, the set of cones $\mathcal{C}$ forms a uniformly definable basis of clopen sets ("uniformly" here means the same formula is used, changing its parameters, to define all basic open sets). We work with the topology generated by this basis which is Hausdorff and totally disconnected.

Definition 4. A C-structure M is *C-minimal* if for every elementary equivalent structure $N \equiv M$, every definable subset $D \subseteq N$ is definable by a quantifier free formula using only the C-predicate. A complete theory is C-minimal if it has a C-minimal model.

Examples 5.

- By quantifier elimination, algebraically closed valued fields are C-minimal with respect to the C-relation defined by the associated ultrametric (i.e., $C(x, y, z) \Leftrightarrow v(x-y) < v(y-z)$). In [HM94] it was proved that C-minimal fields correspond exactly to algebraically closed valued fields. If M is a C-minimal field, for all $a \in T$ we can identify the set of cones at a with the residue field. Since the residue field is algebraically closed, we have in particular that $bn(a) \geq \aleph_0$ for all $a \in T$.
- By a result of Lipshitz and Robinson in [LR98], algebraically closed valued fields enriched with all strictly convergent analytic functions are C-minimal.
- Also by quantifier elimination the additive group of the p-adic field is C-minimal. Though the p-adic field is not.

We now study basic properties of C-minimal structures and assume from now on that M is a C-minimal structure M in a language L. By axioms $(C1) - (C4)$ below presented, cones and 0-level sets form a directed family, that is, for $B_1, B_2 \in \mathcal{C} \cup \mathcal{L}_0$ one of the following holds:

$$B_0 \subseteq B_1 \qquad B_1 \subseteq B_0 \qquad B_0 \cap B_1 = \emptyset.$$

By C-minimality every $L(M)$-formula $\phi(x)$ where $|x| = 1$ is equivalent to a boolean combination of formulas of the form $C(\alpha_1, x, \alpha_2)$ and $\neg C(x, \alpha_1, \alpha_2)$, which respectively define cones and 0-level sets. For instance, finite definable sets $\{\alpha_1, ..., \alpha_n\}$ correspond to the union of 0-level sets $\neg C(x, \alpha_i, \alpha_i)$ for $1 \leq i \leq n$ and M itself to $\{x \in: M \models \neg C(\alpha, x, \alpha)\}$. A *Swiss cheese* S is a set of the form $B_0 \setminus (B_1 \cup \cdots \cup B_n)$ where each B_i is a cone or 0-level set, $B_i \subset B_0$ for all $0 < i$ and $B_i \cap B_j = \emptyset$ for all $0 < i < j$; sets $B_1, ..., B_n$ are called the holes of S. It is not difficult to prove that every definable set defined by a boolean combination of cones and 0-level sets can be expressed as a disjoint union of Swiss cheeses (see [Hol95]). An application of the compactness theorem implies then the following lemma:

Lemma 6. *Let $\phi(x, y)$ be an L-formula with $|x| = 1$. There exist positive integers n_1 and n_2 such that for all $\alpha \in M^{|y|}$, $\phi(M, \alpha)$ is equal to a disjoint union of at most n_1 Swiss cheeses each with at most n_2 holes.*

Proof: Suppose not. By compactness there is an elementary extension $M \prec M_1$ and $\alpha \in M_1^{|y|}$ such that $\phi(M_1, \alpha)$ is not a finite disjoint union of Swiss chesses, which contradicts C-minimality. □

In chapter 2 we will refine the previous lemma showing that every definable unary subset of M can be uniformly decomposed into a finite disjoint union of points, cones, intervals and n-level sets. This will be part of the cell decomposition theorem. Given a set N interpretable in M, that is, a set which correspond to M^n/E for E a definable equivalence relation, the *induced structure* on N corresponds to the structure N together with all relations of cartesian powers of N which are interpretable in M. In many cases, model theoretic properties from M impose model theoretic properties on their induced structures. We show some examples.

Definition 7.

1. Let $\alpha \in M$. There is a definable equivalence relation R_α on M defined by

$$R_\alpha(\beta_1, \beta_2) \Leftrightarrow \inf(\beta_1, \alpha) = \inf(\beta_2, \alpha).$$

   The structure $Br(\alpha)$ (the branch of α is the induced structure by M on M/R_α.

2. For $\alpha, \beta \in M$ and $a = \inf(\alpha, \beta)$, the structure $\mathcal{C}(a)$ is the induced structure by M on M/E_a. It corresponds to the induced structure on the set of cones at a.

The structure $Br(\alpha)$ is isomorphic as an order to $\alpha_{\leq}$. Thus, in an ultrametric space, the order $Br(\alpha)$ is isomorphic to dM, for every $\alpha \in M$. It is worth noticing that not every branch of $T(M)$ has a leaf. In particular, in an ultrametric space M, branches of $T(M)$ can be seen as pseudo-Cauchy sequences (which do not necessarily have a limit in M). Recall an ordered structure $(N, \leq, ...)$ is o-minimal is every definable subset $D \subseteq N$ is a finite union of intervals and points. A structure N is *strongly minimal* if every definable subset $D \subseteq N$ is either finite or cofinite (in all elementary equivalent structures). A consequence of lemma 6 that will be often used is:

Lemma 8. *The structure $Br(\alpha)$ is o-minimal for every $\alpha \in M$. For all $a \in T$, the structure $\mathcal{C}(a)$ is either finite or strongly minimal.*

Proof: Suppose towards a contradiction that there is a definable subset D of $Br(B)$ which is not a finite union of points and intervals. Let D' be the union of all cones $\Gamma_{\inf(\alpha,\beta}(\alpha)$ for $\inf(\alpha, \beta) \in D$. It is not difficult to see that D' cannot be a finite union of Swiss chesses which contradicts C-minimality. Analogously, let D be a definable infinite and coinfinite subset D of $\mathcal{C}(a)$. Then the set D' defined as the union of all cones $\Gamma_a(\alpha)$ such that $\alpha/E_a \in D$ cannot be a finite union of Swiss cheeses, contradicting C-minimality. □

Compare the previous lemma to the fact that in an algebraically closed valued field, the value group is o-minimal and the residue field is algebraically closed, hence strongly minimal. In proofs, we will usually not make explicit reference to lemma 8 but use expressions like “by o-minimality of the branch...” or “by strong minimality of the set of cones at a”, etc. We use two lemmas stated without a proof in [HM94] and some corollaries that will be later used (we provide a proof for lemmas 9 and 10 which correspond to facts 1 and 2 in [HM94] in the appendix).

Lemma 9. *Let $D \subseteq M$ be a definable set. Then, there is no $\alpha \in M$ such that for an infinite number of nodes $a < \alpha$ we have both*

$$\Lambda_a(\alpha) \cap D \neq \emptyset \text{ and } \Lambda_a(\alpha) \cap (M \setminus D) \neq \emptyset. \tag{1}$$

Lemma 10. *Let $D \subseteq M$ be a cone and $f : D \to T$ be a definable function such that $f(\alpha) \in Br(\alpha)$ for all $\alpha \in D$. Then there are no arbitrarily large sequences $\alpha = (\alpha_i : i \leq N)$ and $\beta = (\beta_i : i < N)$ satisfying $\phi_N(\alpha_0, \dots, \alpha_N, \beta_0, \dots, \beta_{N-1})$ defined by*

$$\bigwedge_{i=0}^{N-1} f(\beta_i) \notin Br(\alpha_N) \wedge \bigwedge_{i=0}^{N-1} \inf(\alpha_i, \alpha_N) = \inf(\beta_i, \alpha_N) > f(\alpha_i).$$

Next lemmas will give us different uniform bounds that will be later used for the proof of the cell decomposition theorem.

Lemma 11. *For every L-formula $\phi(x, y)$ with $|x| = 1$, there is a positive integer N such that for all $a \in T$ and all $\alpha \in M^{|y|}$ either $|\{\Gamma_a \in \mathcal{C} : \Gamma_a \subseteq \phi(M, \alpha)\}| < N$ or $|\{\Gamma_a \in \mathcal{C} : \Gamma_a \subseteq \neg\phi(M, \alpha)\}| < N$.*

Proof: Suppose there is no such N. By compactness, there are an elementary extension M_1 of M, $a \in T(M_1)$ and $\alpha \in M_1^{|y|}$ such that both $|\{\Gamma_a \in \mathcal{C} : \Gamma_a \subseteq \phi(M_1, \alpha)\}| \geq \aleph_0$ and $|\{\Gamma_a \in \mathcal{C} : \Gamma_a \subseteq \neg\phi(M_1, \alpha)\}| \geq \aleph_0$ which contradicts the strong minimality of the set of cones at a in M_1. □

Lemma 12. *For every formula $\phi(x, y)$ with $|x| = 1$ there is a positive integer N such that for all $\alpha \in M^{|y|}$ and all $\beta \in M$ the set*

$$A = \{a \in Br(\beta) : \Lambda_a(\beta) \cap \phi(M, \alpha) \neq \emptyset \text{ and } \Lambda_a(\beta) \cap \neg\phi(M, \alpha) \neq \emptyset\}$$

has cardinality less than N.

Proof: If not, by compactness we get α and β in an elementary extension M_1 of M such that A if infinite. This contradicts lemma 9. □

Lemma 13. *For every formula $\phi(x, y)$ with $|x| = 1$ there is a positive integer N such that for all $\alpha \in M^{|y|}$ and all $\beta \in M$, the cardinality of the set points in $Br(\beta)$ which are ending points of intervals maximally contained either in $\phi(M, \alpha)$ or in $\neg\phi(M, \alpha)$ is less than N.*

Proof: If not, by compactness we get a definable infinite discrete subset of $Br(\beta)$ for $\beta \in M_1$ and M_1 an elementary extension of M. This contradicts o-minimality of $Br(\beta)$. □

Assuming density, every cone in a C-structure is infinite. As a consequence dense C-minimal structures can distinguish between finite and infinite definable sets. We summarize in the following lemma:

Lemma 14. *Suppose M is dense. Then*

1. *Every cone is infinite.*
2. *For each L-formulas $\phi(x, y)$ with $|x| = 1$, there is a positive integer N_ϕ such that for all $\alpha \in M^{|y|}$, if $|\phi(M, \alpha)| > N_\phi$ then $|\phi(M, \alpha)| \geq \aleph_0$.*

Proof: For 2 see [HM94] lemma 2.4. Point 1 follows directly from the density axiom (assuming M is not just one point). [1] □

Notice that an ultrametric space M is dense (as a C-set) if and only if the order $dM \setminus \{\infty\}$ has no maximal element.

2 Cell decomposition revisited

Heuristically, the aim of a cell decomposition theorem is to have a general description of definable sets as unions of some special -and hopefully simple- definable sets called *cells*. Usually one can also imply from it that a given dimension function behaves well for definable sets. In most cases, cells are defined by induction: given a structure M, one selects first a collection $\{D_i \subseteq M : i \in I\}$ of definable subsets of M to be the family of 1-cells and then defines by induction n-cells –commonly using definable functions– which correspond to definable subsets of M^n. We start this chapter defining what cells are, discussing alternative definitions with examples. Later, we prove a uniform version of 1-cell decomposition which gives a rough version of cell decomposition for C-minimal structures (not necessarily dense). Then, assuming density, we study definable functions and prove, in analogy with o-minimality, a monotonicity theorem for dense C-minimal structures. Essentially we will prove that definable functions are "cellwise" continuous, that is, we can always decompose their domain into finitely many cells on which the function is continuous. [2]

2.1 Cells

As before, we work in a C-minimal structure M in a language L. We start showing how to define an induce C-relation on antichains of $T(M)$. Let $S \subseteq T(M)$ be an antichain and let $T[S]$ be the closure of S under inf. It is easy to check that $T[S]$ is a good tree, so the set of leaves in $T[S]$ (which corresponds to S) is a C-set where the C-relation is given by $C(x, y, z) \Leftrightarrow \inf(x, y) < \inf(y, z)$. The C-set S having $T[S]$ as its canonical tree is denoted by $M[S]$. Since M can be identified with the set of leaves of $T(M)$, and any $B \subseteq M$ forms an antichain in $T(M)$, we use the expressions $T[B]$ and $M[B]$ viewing B as a subset of $T(M)$. Notice that with this notation we have that $T(M) = T[M]$. We define now what 1-cells are. Given a set A, we denote by $A^{[r]}$ the set of all subsets of A of size exactly r.

Definition 15 (1-cells)**.** Let D be a definable subset of M and L_0 be the language containing only the predicate C.

(I) D is a 1-cell of type $M^{[r]}$ if there is $\{\alpha_1, ..., \alpha_r\} \in M^{[r]}$ such that:

(a) $D = \{\alpha_1, ..., \alpha_r\}$;

(b) $Aut_{L_0}(M[D])$ acts transitively on D.

1. Property (2) is often phrased as the elimination of the quantifier $\exists^\infty$.
2. All results were first proved in [HM94], though in the present exposition some definitions differ and proofs have been changes accordingly.

(II) D is a 1-cell of type $\mathcal{C}^{[r]}$ if there is $\{H_1, ..., H_r\} \in \mathcal{C}^{[r]}$ such that:

(a) $D = \bigcup_{i=1}^{r} H_i$;

(b) the set of bases $A := \{\min(H_i) : 1 \leq i \leq r\}$ is an antichain;

(c) there is a positive integer k such that for all $a \in A$ the set $\{H_i : \min(H_i) = a, 1 \leq i \leq r\}$ has cardinality k;

(d) $Aut_{L_0}(M[A])$ acts transitively on A.

(III) D is a 1-cell of type $\mathcal{L}_n^{[r]}$ $(n < \omega)$ if there is $\{H_1, ..., H_r\} \in \mathcal{L}_n^{[r]}$ such that

(a) $D = \bigcup_{i=1}^{r} H_i$;

(b) the set of bases $A := \{\min(H_i) : 1 \leq i \leq r\}$ is an antichain of cardinality r;

(c) $Aut_{L_0}(M[A])$ acts transitively on A.

(IV) D is a 1-cell of type $I^{[r]}$ if there is $\{I_1, ..., I_r\} \in \mathcal{I}^{[r]}$ such that

(a) $D = \bigcup_{i=1}^{r}\{I, ..., I_r\}$,

(b) the set of left end-points $A := \{a \in T(M) : a \text{ is a left end-point of } I_j \text{ for } 1 \leq j \leq r\}$ is an antichain;

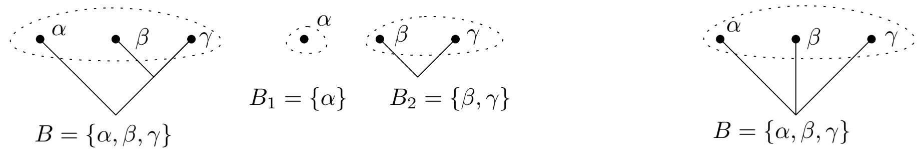
(c) there is a positive integer k such that for all $a \in A$, $|\{I_j \in \{I_1, ..., I_r\} : a \text{ is a left end-point of } I_j\}| = k$;

(d) $I_i \cap I_j = \emptyset$ for all $1 \leq i < j \leq r$;

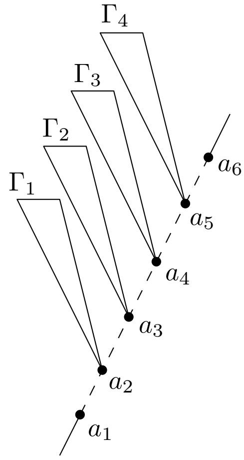
(e) $Aut_{L_0}(M[A])$ acts transitively on A.

It is important to notice that a 1-cell can be of different types. For example a finite 1-cell $B = \{\alpha_1, ..., \alpha_r\}$ is of type $M^{[r]}$ but also of type $\mathcal{L}_0^{[r]}$ since singletons are degenerated 0-level sets. More problematic, in a C-structure M having an element $a \in T(M)$ such that $bn(a) = 2$, the 1-cell $B := \Lambda_a$ is both of type $\mathcal{L}_0^{[1]}$ and of type $\mathcal{C}^{[2]}$ since it is also the union of the two cones at a. We exhibit some examples.

Example 16. Let $B := \{\alpha, \beta, \gamma\}$ be a definable set of type $M^{[3]}$. Suppose that $C(\alpha, \beta, \gamma)$ holds as in the figure on the left. Then $\{\alpha\}$ is an orbit of $Aut_{L_0}(B)$ hence B can be decomposed as the union of two 1-cells $B_1 := \{\alpha\}$ and $B_2 := \{\beta, \gamma\}$. If in contrast we suppose that there is no C-relation between α, β and γ, then B is a 1-cell (figure in the right).

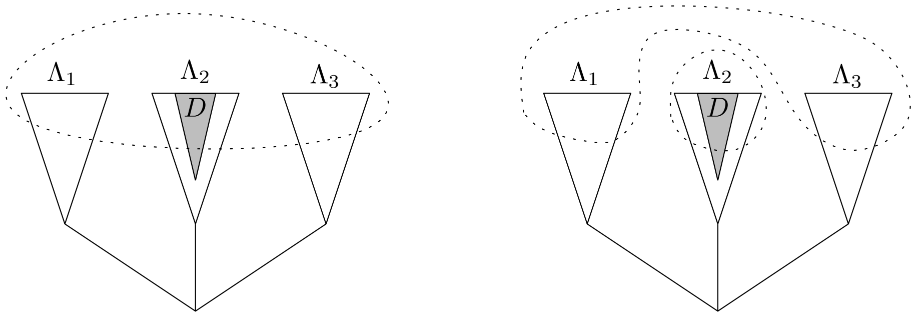


Example 17. Let $a_1, ..., a_6 \in T$ such that a_i is the predecessor of a_{i+1} for all $1 \leq i \leq 5$. Suppose in addition that $bn(a_i) = 2$ for all $1 \leq i \leq 5$. Let $B := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, where each Γ_i is a cone for $1 \leq i \leq 4$ as shown in the figure (in figures, points are connected by a dotted line if there is no point between them). On the one hand, since $B = \Gamma_{a_1}(a_2) \setminus \Lambda_{a_6}$, it is a 1-cell of type $\mathcal{I}^{[1]}$. On the other hand, B is also a set of type $\mathcal{C}^{[4]}$ and as such it can be decomposed into four 1-cells of type $\mathcal{C}^{[1]}$ since their bases form a chain.



In [HM94] the definition of 1-cell involves a notion of irreducibility which seems to depends on the ambient language L (see p. 119-120). The definition of 1-cell here defined has a similar version but with respect to the minimal language common to all C-minimal structures, namely $L_0 = \{C\}$. To get tider definitions of 1-cell one can modify definition 15 letting enrich the language L_0 in the automorphism group Aut_{L_0} of the C-structures considered. Nevertheless, if no assumption whatsoever is made about this enriched language we may have problems concerning uniformity (see example 24). We give a very basic example of how to get tider notions of cells.

Example 18. Suppose L contains a predicate D which is interpreted in M as a cone. Suppose furthermore the automorphism group we look at in definition 15 is defined with respect to the language $L_0 = \{C, D\}$. Consider a set $B = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ of type $\mathcal{L}_0^{[3]}$ such that $D \subseteq \Lambda_2$ as in the figure below. Let $A = \{\min(\Lambda_1), \min(\Lambda_2), \min(\Lambda_3)\}$. Then $\min(\Lambda_2)$ is an orbit of $Aut_{L_0}(M[A])$, so B can be decomposed into two 1-cells, $B_1 = \Lambda_1 \cup \Lambda_3$ and $B_2 = \Lambda_2$. In contrast, B is a 1-cell if we take $L_0 = \{C\}$.



All 1-cells are in particular disjoint unions of Swiss cheeses, but given a 1-cell of a specified type we can recover the elements defining the cell (which is not always the case with a Swiss cheese). This is the content of the following lemma.

Lemma 19. *Let D be a 1-cell of type $Z^{[r]}$. Then there is only one element in $Z^{[r]}$ satisfying all properties of definition 15 for D.*

Proof: Suppose towards a contradiction there are two different elements $\{H_1, ..., H_r\}$ and $\{K_1, ..., K_r\}$ in $Z^{[r]}$ satisfying all properties in definition 15 for D. We split in cases depending on the value of Z.

- **Case** $Z = M$**:** In this case, condition (a) already implies $\{H_1, ..., H_r\} = \{K_1, ..., K_r\}$.

• **Case** $Z = \mathcal{C}$**:** By assumption, there is $1 \leq i \leq r$ such that $H_i \neq K_j$ for all $1 \leq j \leq r$. Be renumbering we may assume $i = 1$. By condition (a) and renumbering if necessary, we may assume that $H_1 \cap K_1 \neq \emptyset$. Thus, either $H_1 \subsetneq K_1$ or $K_1 \subsetneq H_1$. Respectively for each case, there is $1 < i \leq r$ such that $H_i \cap K_1 \neq \emptyset$ or $K_i \cap H_1 \neq \emptyset$. Notice that $H_i \cap H_j = \emptyset$ for all $0 < i < j \leq r$ by condition (b). Suppose without loss of generality the former happens and by renumbering that $i = 2$. Let $A := \{\min(H_i) : 1 \leq i \leq r\}$, $B := \{\min(K_i) : 1 \leq i \leq r\}$ and k_1, k_2 be the positive integers from condition (c) respectively for $\{H_1, ..., H_r\}$ and $\{K_1, ..., K_r\}$. Notice that condition (a) implies that every element in A is comparable to an element of B and viceversa. Since A is an antichain, $H_2 \subseteq K_1$. Therefore, given that $\min(K_1) < \min(H_1)$ and $\min(K_1) < \min(H_2)$, there must be an element $a \in A$ such that $a < b$ for some $b \in B$, otherwise we contradict the fact that both sets of cones have r elements. We split in two cases. Suppose first that $\min(H_1) \neq \min(H_2)$. In this case, since B is an antichain we have that $C(a, \min(H_1), \min(H_2))$, which contradicts condition (d). So suppose that $\min(H_1) = \min(H_2)$. This implies that $k_1 > 1$, therefore there are $i, j \in \{3, ..., n\}$, $i \neq j$ such that $\min(H_i) = \min(H_j) = a$ and $H_i = \Gamma_a(b)$. Hence, there is $b' \in B$ such that $\inf(b, b') = a$. Since A is an antichain, this implies $C(\min(K_1), b, b')$ which contradicts again condition (d).

• **Case** $Z = \mathcal{L}_0$**:** As before let $A := \{\min(H_i) : 1 \leq i \leq r\}$ and $B := \{\min(K_i) : 1 \leq i \leq r\}$. As in the previous case, we may assume $H_1 \subsetneq K_1$, $H_2 \subsetneq K_1$ and that there is $a \in A$ such that $a < b$ for some $b \in B$. Here by condition (b) we already have that $\min(H_1) \neq \min(H_2)$ and therefore since A is an antichain we have that $C(a, \min(H_0), \min(H_1))$, which contradicts condition (c) (of III).

• **Case** $Z = \mathcal{L}_n$ $(n > 0)$**:** Take A and B as before. Notice that if $A = B$, condition (b) already implies the result. So suppose for a contraction that $A \neq B$. By (a) and possibly renumbering we may suppose that $\min(H_1) < \min(K_1)$ and $K_1 \subsetneq H_1$. But then, given that $n > 0$, there is at least one cone at $\min(K_1)$ which is not contained in K_1 but contained in H_1. Therefore there must be some $1 < i \leq r$ such that K_i contains that cone, but then its base will be comparable with $\min(K_1)$ which contradicts condition (b).

• **Case** $Z = I$**:** Let $\Gamma_i \setminus \Lambda_i = H_i$ and $\Gamma'_i \setminus \Lambda'_i = K_i$ for all $1 \leq i \leq r$, $A := \{\min(\Gamma_i) : 1 \leq i \leq r\}$, $B := \{\min(\Gamma'_i) : 1 \leq i \leq r\}$ and k_1, k_2 be the positive integers from condition (c) respectively for $\{H_1, ..., H_r\}$ and $\{K_1, ..., K_r\}$. As before, we may assume $H_1 \cap K_1 \neq \emptyset$ and $H_1 \neq K_1$. Therefore $\min(\Gamma_1)$ and $\min(\Gamma'_1)$ are comparable. We split in two cases. Suppose first that $\min(\Gamma_1) = \min(\Gamma'_1)$. Without loss of generality suppose that there is $x \in H_1 \setminus K_1$, so by condition (a) there is $j \neq 1$ such that $H_1 \cap K_j \neq \emptyset$. Hence, $\min(\Gamma'_1)$ and $\min(\Gamma'_j)$ are comparable, which since B is an antichain implies they are equal. By (d), $K_1 \cap K_j = \emptyset$, but this implies that either Γ'_1 or Γ'_j does not intersect Γ_1, a contradiction. So suppose $\min(\Gamma_1) \neq \min(\Gamma'_1)$. Note that conditions (b), (c) and (e) imply that both $\bigcup_{i=1}^r \Gamma_i$ and $\bigcup_{i=1}^r \Gamma_i$ are 1-cells of type $\mathcal{C}^{[r]}$, so a similar argument as in case $Z = \mathcal{C}$ applies here. □

To define n-cells we need to provide topologies for all $Z^{[r]}$ where Z is one of $M, T, \mathcal{C}, \mathcal{I}$ and $\mathcal{L}_n$ for $n < \omega$. Given a basis $\mathcal{B} = \{U_i : i \in I\}$ for a topology in Z $(= Z^{[1]})$, the topology on $Z^{[r]}$ for $r > 1$ is given by the following basis: for $(i_1, \ldots, i_r) \in I^r$, a basic open set of $Z^{[r]}$ is a set of the form $\{\{a_{i_1}, \ldots, a_{i_r}\} : a_{i_j} \in U_{i_j}, 1 \leq j \leq r\}$ (allowing here $i_j = i_{j'}$ for $j \neq j'$). By lemma 2.1 in [HM94], if the topology on Z has a uniformly definable subbasis, then for any positive integer r the topology on $Z^{[r]}$ has a uniformly definable subbasis. Therefore we are left with the definitions of topologies on $M, T, \mathcal{C}, \mathcal{I}$ and $\mathcal{L}_n$ for $n < \omega$. On M, as previously stated, we take the set of cones as a uniformly definable basis for its topology. For the rest we take what could be called interval topologies. For $\Gamma \in \mathcal{C}$ and $\Lambda \in \mathcal{L}_0$ a subbasic open of T is defined by

$$(\Gamma, \Lambda)_T := \{x \in T : x \in \Gamma \setminus \Lambda\}.$$

Notice that $\min(\Gamma)$ does not belong to (Γ, Λ) by the definition of a cone. Since 0-level sets and points in T are interdefinable we take the topology in on $\mathcal{L}_0$ to be the induced topology on T, that is,

$$(\Gamma, \Lambda)_{\mathcal{L}_0} := \{x \in \mathcal{L}_0 : \min(x) \in (\Gamma, \Lambda)_T\}$$

In both cases, subbasic open sets are uniformly definable. A subbasic open for the topologies on the set of cones $\mathcal{C}$ and the n-level sets for $0 < n < \omega$ corresponds to

$$(\Gamma, \Lambda)_{\mathcal{C}} := \{x \in \mathcal{C} : x \subseteq \Gamma \setminus \Lambda\},$$

$$(\Gamma, \Lambda)_{\mathcal{L}_n} := \{x \in \mathcal{L}_n : x \subseteq \Gamma \setminus \Lambda\}.$$

Notice that for a cone D it is not enough that $\min(D) \in (\Gamma, \Lambda)_T$ to have that $D \in (\Gamma, \Lambda)_{\mathcal{C}}$, since it could be the case that $\min(D) \in (\Gamma, \Lambda)$ but $\Lambda \subseteq D$. The same happens with n-level sets for $n > 0$. Finally, the topology in $\mathcal{I}$ is the topology induced by the product topology in T^2, that is, a subbasic open corresponds to

$$(\Gamma_1, \Lambda_1, \Gamma_2, \Lambda_2) := \{I \in \mathcal{I} : lp(I) \in (\Gamma_1, \Lambda_1)_T, rp(I) \in (\Gamma_2, \Lambda_2)_T)\},$$

where $lp(I)$ is the left end-point of I and $rp(I)$ is the right end-point of I. We are now ready to define n-cells.

Definition 20. Let $Y \subseteq M^n$ be a definable set, $n > 1$ and $\pi : M^n \to M^{n-1}$ be the projection of M^n onto the first $n-1$ coordinates. Y is an *n-cell* of type $(Z_1^{[r_1]}, \ldots, Z_n^{[r_n]})$, where Z_i ranges over $M, \mathcal{C}, \mathcal{I}$ and $\mathcal{L}_m$ ($m < \omega$) and r_i is a positive integer for all $1 \leq i \leq n$, if $\pi(Y)$ is an $(n-1)$-cell of type $(Z_1^{[r_1]}, \ldots, Z_{n-1}^{[r_{n-1}]})$ and either:

1. $Z_n = M$ and $Y = \{(y, z) \in \pi(Y) \times M : z \in f(y)\}$, where $f : \pi(Y) \to M^{[r_n]}$ is a definable continuous function and $f(y)$ is a 1-cell of type $M^{[r_n]}$ for all $y \in \pi(Y)$.
2. $Y = \{(y, z) \in \pi(Y) \times M : z \in \bigcup f(y)\}$, where $f : \pi(Y) \to Z_n^{[r_n]}$ is a definable continuous function, Z_n ranges over $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{L}_m$ ($m < \omega$) and $\bigcup f(y)$ 1-cell of type $Z_n^{[r_n]}$ for all $y \in \pi(Y)$.

Y is an *almost n-cell* if the continuity condition is dropped. A *decomposition* (resp. an almost decomposition) of a definable set $D \subseteq M^n$ is a finite set of disjoint n-cells (resp. almost n-cells) $\{Y_1, \ldots, Y_m\}$ such that $D = \bigcup_{i=1}^m Y_i$. A cell is an n-cell for some positive integer n.

It is easy to show by induction that M^n is a cell and $\alpha \in M^n$ are a cells for all $1 \leq n < \omega$. The following shows how to uniformly decompose a definable subset of M into finitely many 1-cells.

Proposition 21 (Uniform 1-cell decomposition)**.** *Let $\phi(x, y)$ be an L-formula with $|x| = 1$. There is a finite definable partition $\mathcal{P}$ of $M^{|y|}$ such that for each $A \in \mathcal{P}$ there are L-formulas $\psi_1^A(x, y), \ldots, \psi_{n_A}^A(x, y)$ satisfying that whenever $\alpha \in A$, $\{\psi_1^A(M, \alpha), \ldots, \psi_{n_A}^A(M, \alpha)\}$ is a 1-cell decomposition of $\phi(M, \alpha)$. Moreover, each formula $\psi_j^A(x, y)$ defines the same type of 1-cell for each $\alpha \in A$.*

Proof: The proof mimics ideas from proposition 3.7 in [Del11]. For $\alpha \in M^{|y|}$, let

$$T_1(\alpha) := \{a \in T(M) : \exists\beta(\phi(\beta,\alpha) \wedge a < \beta) \wedge \exists\beta(\neg\phi(\beta,\alpha) \wedge a < \beta)\}.$$

Note that $T_1(\alpha) = \emptyset$ if and only if $\phi(M,\alpha) = M$ or $\phi(M,\alpha) = \emptyset$. Let $A_0 := \{\alpha \in M^{|y|} : T_1(\alpha) = \emptyset\}$. Since M is a cone (we treat M as a cone at $-\infty$), setting $n_{A_0} := 1$ and $\psi_1^1(x,y)$ as $\phi(x,y)$ we already have the result for A_0. Take $\alpha \in M^{|y|} \setminus A_0$ (so $T_1(\alpha) \neq \emptyset$) and let $D_\alpha := \phi(M,\alpha)$. From now on, to ease notation we omit reference to α if no ambiguity arises having $D = D_\alpha$, $T_1 = T_1(\alpha)$, etc. Consider the sets

$$\begin{aligned} X &:= \{a \in T(M) : a \text{ is a supremum of a branch in } T_1\}; \\ Y &:= \{a \in T_1 : a \text{ branches in } T_1\}; \end{aligned}$$

Claim 1: There is a positive integer N such that for all $\alpha \in M^{|y|} \setminus A_0$ the cardinality of $X_\alpha \cup Y_\alpha$ is less than N.

Proof of the claim: By lemma 6 there is n such that for all $\alpha \in M^{|y|}$, $\phi(M,\alpha)$ is equivalent to a disjoint union of at most n Swiss cheeses each with at most n holes. For $\alpha \in M^{|y|} \setminus A_0$ let B_α be the set of bases of cones and 0-level sets in a given Swiss cheese presentation of $\phi(M,\alpha)$. Let $T_0(\alpha) := cl_{\leq}(B_\alpha)$. One can easily check that independently of B_α, $T_1(\alpha) \subseteq T_0(\alpha)$. This implies in particular that $T_1(\alpha)$ has finitely many branches (less than n^2) for all $\alpha \in M^{|y|} \setminus A_0$ and thus finitely many branching points. Consequently, the cardinality of both X_α and Y_α is uniformly bounded, which proves the claim.

We define for $a \in T_1 \setminus (X \cup Y)$ an element $c_a = \inf\{x \in X \cup Y : a < x\}$, which is well-defined by the claim. Consider now the sets:

$$\begin{aligned} W &:= \{a \in T_1 \setminus (X \cup Y) : \Lambda_a(c_a) \cap D \neq \emptyset \wedge \Lambda_a(c_a) \cap M \setminus D \neq \emptyset)\} \\ V &:= \{a \in T_1 \setminus (X \cup Y \cup W) : a \text{ is the left-ending point of an interval of } T_1 \\ &\qquad \text{which is maximal for being contained in } \{a \in T_1 \setminus (X \cup Y \cup W) : \Lambda_a(c_a) \subseteq \\ &\qquad D \text{ or } \Lambda_a(c_a) \subseteq M \setminus D\}\}. \end{aligned}$$

X, Y, W and V also depend on α and are uniformly definable. To stress this dependence or clarify an ambiguity we add indices and denote them by $X_\alpha, Y_\alpha, W_\alpha$ and V_α if needed.

Claim 2: There is a positive integer N such that for all $\alpha \in M^{|y|} \setminus A_0$ the cardinality of $F_\alpha = X_\alpha \cup Y_\alpha \cup W_\alpha \cup V_\alpha$ is less than N.

Proof of the claim: By the claim 1, the cardinality of both X_α and Y_α is uniformly bounded. The cardinalities of W_α and V_α are uniformly bounded by lemmas 12 and 13 respectively. This completes claim 2.

For each $\alpha \in M^{|y|} \setminus A_0$ we consider $F_\alpha = X_\alpha \cup Y_\alpha \cup W_\alpha \cup V_\alpha$ as a finite tree structure with predicates for X, Y, W and V. By claim 2, there are finitely many non-isomorphic such tree structures. Hence, there is a definable finite partition $\mathcal{P}$ of $M^{|y|} \setminus A_0$ such that α and α' are in the same element of $\mathcal{P}$ if and only if $F_\alpha \cong F_{\alpha'}$. Fix $A \in \mathcal{P}$ and $\alpha \in A$. Let r be the root of F_α and for $a, b \in F_\alpha$ such that $a < b$, let $a^+(b)$ be the successor of a in the interval $(a, b]$ in F_α. Both r and $a^+(b)$ exist since F_α is finite. Consider

- $Z_1 := \{\Lambda_a \cap D : a \in X\}$;
- $Z_2 := \{\Big(\Lambda_a \setminus \bigcup_{b \in X, a<b} \Gamma_a(b)\Big) \cap D : a \in Y\}$;
- $Z_3 := \{\Lambda_a(c_a) \cap D : a \in W\}$;
- $Z_4 := \{\Lambda_a(c_a) \cap D : a \in V\}$;
- $Z_5 := \{\Big(\Gamma_a(b) \setminus \Lambda_{a^+(b)}\Big) \cap D : a \in Y, b \in X$ such that $a < b\}$;
- $Z_6 := \{(\Gamma_a(c_a) \setminus \Lambda_{a^+}) \cap D : a \in W\}$;
- $Z_7 := \{(\Gamma_a(c_a) \setminus \Lambda_{a^+}) \cap D : a \in V\}$;
- If $T(M)$ has no root, then $Z_8 = \{(M \setminus \Lambda_r) \cap D\}$.

Elements in Z_1 to Z_4 are either cones or n-level sets and elements in Z_5 to Z_8 are intervals. That $D = \bigcup_{j=1}^{8} \bigcup Z_j$ can be checked by cases and is left to the reader. We now decompose into 1-cells. We first deal with the problem of fixing their type of cell. For each $a \in F$ let n_a be the number of cones at a contained in D and m_a the number of cones at a contained in $M \setminus D$. By lemma 11, there is a positive integer N such that for all $\alpha \in M^{|y|} \setminus A_0$ and all $a \in F$, either $n_a < N$ or $m_a < N$. We associate to each $z \in \bigcup_{i=1}^{4} Z_i$ a pair $(n_z, m_z) \in (N \cup \{\infty\})^2$ where for a the base of z, n_z is the number of cones at a contained in z and m_z is the number of cones at a contained in $M \setminus z$. Analogously, for an interval $z \in \bigcup_{i=1}^{4} Z_{4i+1}$ we let l_z to be the number of intervals in $\bigcup_{i=1}^{4} Z_{i+1}$ having the same left-ending point as z. Let $ht(F_\alpha) = h$ and for $i \leq h$ let $F_\alpha(i)$ be the set of nodes in F_α of height i. Clearly $F_\alpha(i)$ is an antichain for each $i \leq h$. Consider the following definable sets:

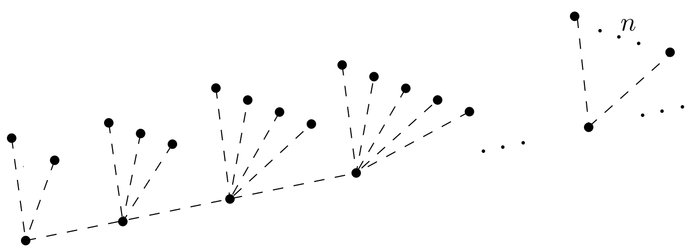
- Let $(n, m) \in (N \cup \{\infty\})^2$ be such that $n = \infty$ or $(n, m) = (0, 0)$. For $i \leq h$, let
$$B(n, m, i) = \{a \in F_\alpha(i) : a = \min(z), (n_z, m_z) = (n, m), z \in \bigcup_{i=1}^{4} Z_i\}.$$
Since $B(n, m, i)$ is an antichain, let $O_1, ..., O_k$ be the orbits of $Aut_{L_0}(M[B(n, m, i)])$ (notice that k is uniformly bounded by $|X|$). Set $\nu_{nmij}(M, \alpha)$ as the union of all $z \in \bigcup_{i=1}^{4} Z_i$ for which $(n_z, m_z) = (n, m)$, $ht(\min(z)) = i$ and $\min(z) \in O_j$ for $j \leq k$. Let s be the number of such elements. By construction, if non-empty, the set $\nu_{nmij}(M, \alpha)$ is a 1-cell of type $\mathcal{L}_m^{[s]}$.
- Let $(n, m) \in (N \cup \{\infty\})^2$ be such that $0 < n \leq N$. For $i \leq h$, let
$$B(n, m, i) = \{a \in F_\alpha(i) : a = \min(z), (n_z, m_z) = (n, m), z \in \bigcup_{i=1}^{4} Z_i\}$$
A before, let $O_1, ..., O_k$ be the orbits of $Aut_{L_0}(M[B(n, m, i)])$. Set $\tau_{nmij}(M, \alpha)$ as the union of all $z \in \bigcup_{i=1}^{4} Z_i$ for which $(n_z, m_z) = (n, m)$, $ht(\min(z)) = i$, $\min(z) \in O_j$ for $j \leq k$. Let s be the number of such elements. By construction, if non-empty, the set $\tau_{nmij}(M, \alpha)$ is a 1-cell of type $\mathcal{C}^{[s]}$.
- For $i \leq h$, and $l < N$ let
$$B(l, i) = \{a \in F_\alpha(i) : a \text{ is the left-endpoint of an element } z \in \bigcup_{i=1}^{4} Z_{4i+1}, l = l_z\}$$
Let $O_1, ..., O_k$ be the orbits of $Aut_{L_0}(M[B(l, i)])$. Set $\tau_{l,i}(M, \alpha)$ as the union of all $z \in \bigcup_{i=1}^{4} Z_i$ for which $(n_z, m_z) = (n, m)$, $ht(\min(z)) = i$, $\min(z) \in O_j$ for $j \leq k$. Let s be the number of such elements. By construction, if non-empty, the set $\xi_{li}(M, \alpha)$ is a 1-cell of type $\mathcal{I}^{[s]}$.

Let $\{\theta_i(x,y) : 0 < i < K_A\}$ be an enumeration of all formulas $\tau_{nmij}(x,y)$, $\nu_{nmij}(x,y)$ and $\xi_{li}(x,y)$ for $(n,m) \in S_\alpha$ and $k,l < N \times |X|$. Then refine $\mathcal{P}$ letting α, α' belong in the same element of $\mathcal{P}$ if and only if $F_\alpha \cong F_{\alpha'}$ and $\theta_i(M,\alpha) \neq \emptyset$ if and only if $\theta_i(M,\alpha') \neq \emptyset$ for all $0 < i < K_A$. Now, for $A \in \mathcal{P}$, let $\{\psi(x,y)_i^A : 0 < i < n_A\}$ be an enumeration of all formulas $\theta_i(x,y)$ such that $\theta_i(M,\alpha) \neq \emptyset$. It is clear that

$$\bigcup_{i=1}^{n_A} \psi_i^A(M,\alpha) = \bigcup_{i=1}^{8} \bigcup Z_i = \phi(M,\alpha).$$

□

Remark 22. It is worth noting that the type of all 1-cells in the above proof correspond only to n-level sets, cones and intervals without having 1-cells of type $M^{[r]}$ (points were treated as 0-level sets). If densityis assumed, by lemma 14 we can further suppose that 1-cells of type $Z^{[r]}$ for Z either $\mathcal{C}, \mathcal{I}$ or $\mathcal{L}_n$ $(n < \omega)$ appearing in the 1-cell decomposition of a definable set are always infinite, letting finite 1-cells to be only of type $M^{[r]}$. This is a simple application of the elimination of the quantifier $\exists^\infty$ (property 2 in lemma 14). Without the elimination of $\exists^\infty$ we could have a C-minimal structure as in the figure below, having a family of arbirary large finite 0-level sets, which implies they could not be defined in a cell decomposition as cells of type $M^{[r]}$ for some positive integer r.



For $\phi(x,y)$ a formula with $|x| = 1$, the set of discrete points in $\phi(M,\alpha)$ is uniformly definable for all $\alpha \in M^{|y|}$ and by C-minimality and density always finite. Then one can provide a cell decomposition of $\phi(M,\alpha)$ where all 1-cells of type $Z^{[r]}$ with $Z \neq M$ contain at least one cone, hence there are infinite.

In order to obtain a uniform cell decomposition we gave up in some cases intuition. For instance, an interval can have different decomposition depending on the type of the parameters. We present an example in valued groups that shows this.

Example 23. Let $L_G = \{+, \text{div}, 0\}$ be the language of valued groups where div is a binary predicate interpreted as

$$div(x,y) \Leftrightarrow v(x) \leq v(y).$$

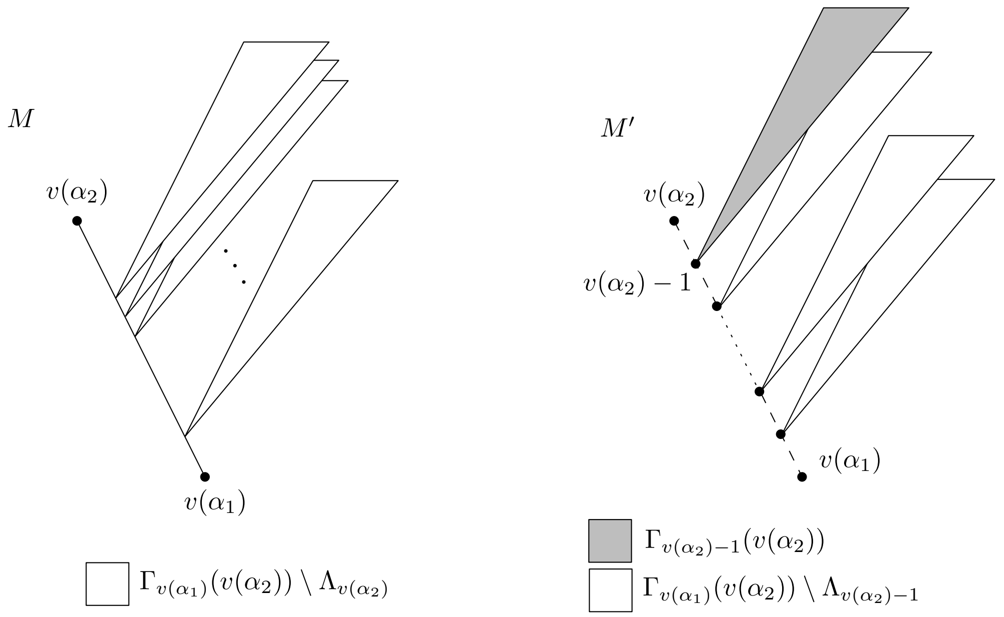
Consider the following L-formula

$$\phi(x, y_1, y_2) := \neg\text{div}(x, y_1) \wedge \neg\text{div}(y_2, x).$$

(A) Let K be an algebraically closed valued field and M its reduct to the language L_G, which is also C-minimal. Here every branch is isomorphic to the value group (which is dense as an order) and we identify points in any branch with points in $v(K)$. Let α denote a tuple (α_1, α_2) and $D := \phi(M,\alpha)$. Proposition 21 gives us a partition $\mathcal{P} = \{A_0, A_1\}$, where $A_0 := \{\alpha \in M^2 : v(\alpha_1) \geq v(\alpha_2)\}$ and $A_1 := M^2 \setminus A_0$. Clearly, $D = \emptyset$ if and only if $\alpha \in A_0$. For $\alpha \in A_1$, F_α is linearly ordered, $X = \{v(\alpha_2)\}$ (where we regard all values like

$v(\alpha_2)$ as elements in the branch of 0), $Y_\alpha = \emptyset$ given that there are no branching points, $W = \emptyset$ and $V_\alpha = \{v(\alpha_1)\}$. The only element in $\bigcup_{i=1}^{8} Z_i$ which is not empty is the element in Z_7 that corresponds to the interval $\Gamma_{v(\alpha_1)}(v(\alpha_2)) \setminus \Lambda_{v(\alpha_2)}$. This shows that $\phi(M, \alpha)$ is already a 1-cell of type $\mathcal{I}^{[1]}$ for all $\alpha \in A_1$.

(B) Consider now M' be the additive group of the 2-adic field in the language L_G. M' is C-minimal. Again, every branch can be identified with the value group but in contrast here the value group is $\mathbb{Z}$ which is discrete. Moreover we have that $bn(a) = 2$ for all $a \in T$. Proposition 21 gives us the same partition $\mathcal{P} = \{A_0, A_1\}$ but with a different 1-cell decomposition. Given that the order is discrete, $X_\alpha = \{v(\alpha_2) - 1\}$. Therefore, for $\alpha \in A_1$, $\phi(M, \alpha)$ is decomposed into the cone at $v(\alpha_2) - 1$ which does not contain $v(\alpha_2)$ (remember $bn(a) = 2$, so this is well-defined) and the interval $\Gamma_{v(\alpha_1)}(v(\alpha_2)) \setminus \Lambda_{v(\alpha_2)-1}$. This shows $\phi(M, \alpha)$ is not a 1-cell in this case.



Next example exhibits a potential problem that might arise by letting L_0 be any sublanguage of the ambient language in the definition of 1-cells (def. 15).

Example 24. Let M be a dense pure C-minimal structure, i.e., in the language $L_0 := \{C\}$. Let X be an infinite discrete subset of M. By C-minimality, X is not definable. Let M' be the expansion of M adding constants for each element of X, i.e., we consider M in the language $L = L_0(X)$. Since C-minimality is preserved by adding constants, M' is also C-minimal. Consider for $y = (y_1, y_2)$ the formula

$$\phi(x, y) := \quad (x = y_1 \vee x = y_2) \wedge y_1 \neq y_2.$$

Suppose 1-cells are defined with respect to the language L instead of L_0 in definition 15 and in addition, towards a contradiction, that proposition 21 is true for this notion of 1-cell. Then there is an L-definable finite partition $\mathcal{P}$ of M^2 such that for each $A \in \mathcal{P}$ there are L-formulas $\psi_1^A(x, y), \ldots, \psi_{n_A}^A(x, y)$ satisfying that whenever $\alpha \in A$, $\{\psi_1^A(M, \alpha), \ldots, \psi_{n_A}^A(M, \alpha)\}$ is a decomposition of $\phi(M, \alpha)$ and each formula $\psi_j^A(x, y)$ defines the same type of 1-cell for each $\alpha \in A$. In this case, there are only two possible decompositions for $\phi(M, \alpha)$ namely, either the union of two 1-cells of type $M^{[1]}$ or a 1-cell of type $M^{[2]}$. For $\alpha = (\alpha_1, \alpha_2)$, if the decomposition of $\phi(M, \alpha)$ is the union of two 1-cells then either α_1 or α_2 must lie in X since this is the only case where they can be in different orbits of $Aut_L([\{\alpha_1, \alpha_2\}])$. Let $A \in \mathcal{P}$ be the subset of M^2 such that for all $\alpha \in A$ the decomposition of $\phi(M, \alpha)$ is the union of two 1-cells. For $\beta \in M \setminus X$, we have that $\{\alpha_1 \in M : (\alpha_1, \beta) \in A\} = X$, which contradicts the fact that X is not definable.

Proposition 21 gives the following rough version of cell decomposition:

Proposition 25. *Every definable set $D \subseteq M^n$ has an almost cell decomposition.*

Proof: The proof goes by induction on n. The case $n = 1$ corresponds to an instance of proposition 21. Let π be the projection onto the first $n-1$ coordinates. Let $\phi(x, y)$ be the formula defining D where $|x| = 1$ and x corresponds to the variable which is dropped by the projection π. For $\alpha \in \pi(D)$ we let $D_\alpha := \{\beta \in M : M \models \phi(\beta, \alpha)\}$. Then by proposition 21, there is a definable finite partition $\mathcal{P}$ of $\pi(D)$ such that for each $A \in \mathcal{P}$, D_α uniformly decomposes into finitely many 1-cells, say $\psi_1^A(x, y), ..., \psi_l^A(x, y)$. By induction, we may assume each $A \in \mathcal{P}$ is already an $(n-1)$-cell. Suppose that $\psi_i^A(x, y)$ defines a 1-cell of type $Z^{[r]}$. Then we have a definable function $h_i^A : A \to Z^{[r]}$ defined by $h_i^A(\alpha) = \psi_i^A(M, \alpha)$. Doing the same for each ψ_i^A and each $A \in \mathcal{P}$ we get functions h_i^A of the desired form. The sets $Y_i^A = \{(\alpha, \beta) \in \pi(A) \times M : \beta \in h_i^A(\alpha)\}$ form a disjoint union of almost n-cells whose union is D. □

Corollary 26. *Let $X \subseteq M^n$ be a definable subset and $X_1, ..., X_s$ definable subsets of X. Then X can be partitioned into finitely many almost n-cells $Y_1, ..., Y_m$ of M^n respecting each X_i, i.e, if $X_i \cap Y_j \neq \emptyset$ then $Y_j \subseteq X_i$.*

Proof: Apply the previous result to each $X \cap \bigcap X_i^{\delta(i)}$ where $\delta \in^s 2$ and $X_i^1 := X_i$ and $X_i^0 := M \setminus X_i$ for all $1 \leq i \leq s$. □

2.2 Definable functions and "cellwise" continuity

Through this section M will be a dense C-minimal L-structure. We aim to prove the following theorem (all terms to be defined):

Theorem 27 (Haskell and Macpherson)**.** *Let Z be one of $M, T, \mathcal{C}, \mathcal{I}$ or $\mathcal{L}_n$ for $n < \omega$. Let r be a positive integer and $f : M \to Z^{[r]}$ be a definable partial function. Then there is a decomposition of dom(f) into finitely many cells on which f is continuous. On each infinite cell, f is reducible to a family of definable functions each of which is either a local multi-isomorphism or locally constant.*

This is an analogous version of the o-minimal monotonicity theorem (see [PS86, KPS86]) for C-minimal structures. To prove it one has to deal with the case $r = 1$ first and then generalise it to multi-functions. We start with some definitions.

Definition 28. Let M and N be two C-structures. A partial function $f : M \to N$ is a C-isomorphism if it is injective and preserves the C-relation. It is a *local C-isomorphism* if for every $\alpha \in \text{dom}(f)$ there is a cone $D \subseteq M$ such that $\alpha \in D \subseteq \text{dom}(f)$ and $f \restriction D$ is a C-isomorphism.

Proposition 29. *[Haskell-Macpherson] Let $f : M \to T(M)$ be a definable partial function. Then, dom(f) can be written as a definable disjoint union of sets $F \cup I \cup K$ where F is finite, f is a local C-isomorphism on I and f is locally constant on K.*

Proof: We present only a sketch of the proof. Two cases are distinguished. First, one supposes that the image of the function is an antichain (this covers already the case $im(f) \subseteq M$). The key idea here is to consider for each $\alpha \in \mathrm{dom}(f)$ a uniformly definable partial function ϕ_α with domain $Br(\alpha)$ and range $Br(f(\alpha))$ as follows:

$$\phi_\alpha(a) = b \text{ if and only if } f(\Lambda_a(\alpha)) \subseteq \Lambda_b(f(\alpha)).$$

One shows that for a fixed $\alpha \in \mathrm{dom}(f)$ the function ϕ_α is well-defined in a cofinite subset of $\{\inf(\alpha, \beta) : \beta \in dom(f)\}$. Once this is proved, by density, the set

$$D := \{\alpha \in \mathrm{dom}(f) : \{\inf(\alpha, \beta) : \beta \in \mathrm{dom}(f)\} \text{ is cofinal in } \alpha\}$$

is cofinite, so one may assume $\mathrm{dom}(f) = D$. Then, by the monotonicity theorem for o-minimal structures (remember by lemma 8 $Br(\alpha)$ is o-minimal), for each function ϕ_α we have that ϕ_α is either monotonically increasing, monotonically decreasing or constant on a cofinal segment of α, so one partitions $\mathrm{dom}(f)$ into subsets

- $P = \{\alpha \in \mathrm{dom}(f) : \phi_\alpha$ is monotonically increasing on a final segment of $Br(\alpha)\}$
- $R = \{\alpha \in \mathrm{dom}(f) : \phi_\alpha$ is monotonically decreasing on a final segment of $Br(\alpha)\}$
- $K = \{\alpha \in \mathrm{dom}(f) : \phi_\alpha$ is constant on a final segment of $\alpha\}$.

One is able to show then that R is finite, f is a local C-isomorphism on a cofinite subset of P and locally constant on a cofinite subset of K. To deal with the general case, one partitions $\mathrm{dom}(f)$ into the following sets (for $a, b \in T(M)$ we use $a \parallel b$ to say that a and b are incomparable):

- $J_1 = \{\alpha \in \mathrm{dom}(f) :$ there is a cone D containing α s.t. $\forall \beta \in D \setminus \{\alpha\}(f(\beta) > f(\alpha))\}$;
- $J_1 = \{\alpha \in \mathrm{dom}(f) :$ there is a cone D containing α s.t. $\forall \beta \in D \setminus \{\alpha\}(f(\beta) < f(\alpha))\}$;
- $J_3 = \{\alpha \in \mathrm{dom}(f) :$ there is a cone D containing α s.t. $\forall \beta \in D \setminus \{\alpha\}(f(\beta) \parallel f(\alpha))\}$;
- $J_4 = \{\alpha \in \mathrm{dom}(f) :$ there is a cone D containing α s.t. $\forall \beta \in D \setminus \{\alpha\}(f(\beta) = f(\alpha))\}$;
- $J_5 = \mathrm{dom}(f) \setminus J_1 \cup J_2 \cup J_3 \cup J_4$.

By density, J_5 is finite. The proof is completed showing that J_1 and J_2 are finite (an argument by contradiction building sequences as in lemma 10) and that f is a local C-isomorphism on J_3 (notice that f is locally constant by definition on J_4). □

We give a more detailed proof of the fact that every definable function is continuous almost everywhere.

Lemma 30. *Let $f : M \to Z$ for Z either $M, T, \mathcal{C}, \mathcal{I}$ or $\mathcal{L}_n$ for $n < \omega$ be a definable partial function. Then f is continuous on a cofinite subset of $dom(f)$.*

Proof: The set $A := \{\alpha \in \mathrm{dom}(f) : f$ is not continuous at $\alpha\}$ is definable in all cases given that all topologies have a uniformly definable subbasis. Suppose towards a contradiction that A is infinite. Then it contains a cone E. We split the proof by cases depending on Z.

Case $Z = M$: By proposition 29, $\mathrm{dom}(f)$ decomposes into sets $F \cup I \cup K$ where F is finite, f is a local C-isomorphism on I and locally constant on K. Being locally constant already implies continuity, so without loss of generality may assume that f is a C-isomorphism on E. Therefore $f(E)$ is infinite, hence contains a cone which contradicts that $E \subseteq A$.

Case $Z = T$: Again by proposition 29 we are reduce to the case where f is a C-isomorphism on E. Take $\alpha \in E$. For $\alpha \in E$, given that $E \subseteq A$, there is an open set $(\Gamma, \Lambda)_T$ containing $f(\alpha)$ such that for all cones H containing α we have that $f(H) \nsubseteq (\Gamma, \Lambda)_T$. In particular, for E, given that $f \restriction E$ is a C-isomorphism, this implies there is $\beta \in E$ such that $f(\beta) \notin (\Gamma, \Lambda)_T$ hence $f(\beta) \in \Lambda$. Given that $f(E)$ is an antichain, $\inf(\min(\Lambda), f(\alpha)) \neq f(\alpha)$, since otherwise $f(\alpha) < f(\beta)$. Thus there is $b > f(\alpha)$ such that $\Lambda_b \subseteq (\Gamma, \Lambda)$. By density, there is also some $\gamma \in E$ such that $C(\beta, \alpha, \gamma)$. Then the open set

$$(\Gamma'\Lambda')_T := (\Gamma_{\inf(f(\gamma), f(\alpha))}(f(\alpha)), \Lambda_b)_T$$

contains $f(\alpha)$ and is contained in (Γ, Λ), therefore cannot contain $f(H)$ for any cone H containing α. This is true in particular, for the cone $\Gamma_{\inf(\alpha,\gamma)}(\alpha) \subseteq E$. Hence there is δ such that $f(\delta) \in \Lambda_b$, but then $f(\alpha) < b \leq f(\delta)$ which contradicts the fact that $f(E)$ is an antichain.

Cases $Z = \mathcal{C}$ and $Z = \mathcal{L}_n$: Functions from M to $\mathcal{L}_0$ are in bijection with functions to T, so we may assume that $0 < n < \omega$. We prove in this case that f is locally constant in a cofinite subset of $\mathrm{dom}(f)$, which implies continuity. Let $\hat{f} : M \to T$ be the induced function defined by $\hat{f}(\alpha) := \min(f(\alpha))$. By proposition 29, $\mathrm{dom}(\hat{f})$ is decomposed into sets $F \cup I \cup K$ where F is finite, $\hat{f}$ is a local C-isomorphism on I and locally constant on K. Take $\alpha \in K$ and let $D \subseteq K$ be a cone containing α such that $\hat{f}$ is constant on D. Notice that the set of cones contained in K and containing α is totally ordered (and can be identified with a cofinal subset of $Br(\alpha)$). Hence by lemma 9 and strong minimality of the set of cones at $\hat{f}(\alpha)$, f must be constant on some cone containing α. To finish the proof, we show I is empty. For suppose not and let D be a cone on which $\hat{f}$ is a C-isomorphism. Consider the following definable set

$$X := \begin{cases} f(D) & \text{if } Z = \mathcal{C} \\ \bigcup\{x \in \Lambda_{\min(f(\alpha))} \setminus f(\alpha) : \alpha \in D\} & \text{if } Z = \mathcal{L} \text{ for } 0 < n < \omega. \end{cases}$$

In both cases, given that $\hat{f}$ is a C-isomorphism on D, X is the union of infinitely many cones having their basis at the antichain $\hat{f}(D)$ and satisfying for all $a \in \hat{f}(D)$ there is $\beta \in \Lambda_{\hat{f}(\alpha)} \setminus X$ (here we use the fact that $n \neq 0$). But by the proof of theorem 21, the tree

$$T(X) := \{a \in T : \exists\beta(\beta \in X \wedge a < \beta) \wedge \exists\beta(\beta \in M \setminus X \wedge a < \beta)\},$$

has finitely many branches, which contradicts the fact that $\hat{f}(D)$ is an infinite antichain.

Case $Z = I$: By the definition of the topology on $\mathcal{I}$, the almost everywhere continuity of f is reduced here to the case $Z = T$ since it follows by the almost everywhere continuity of the induced functions $f_l : M \to T$ and $f_r : M \to T$ sending an element α respectively to the left and right end-points of the interval $f(\alpha)$. □

We give an application that will be heavily used:

Lemma 31. *Let I be a totally ordered subset of $T(M) \cup \{-\infty\}$ with no greatest element and $f : M^n \to T(M)$ be a definable partial function with $rng(f) \subseteq I$. Then, for all $\alpha \in M^n$ there are $b \in I$ and a basic neighborhood U of α such that for all $x \in U$ if $f(x)$ is defined then $f(x) < b$.*

Proof: The proof goes by induction on n.

To prove cell decomposition we need to extend the result to multi-functions. We need first to define what is the corresponding notion of C-isomorphism in this cases, which leads to the concept of multi-isomorphism. Recall that for a function $f : M \to Z^{[r]}$ and $D \subseteq M$, $f(D)$ denotes $\bigcup\{f(x) : x \in D\}$ and we call r the multi-degree of f.

Definition 32. Let M be a C-set and r be a positive integer.

a) A definable partial function $f : M \to T(M)^{[r]}$ is a *strong multi-isomorphism* if $S := \mathrm{im}(f)$ is an antichain and there are cones $B_1, \dots, B_r$ in $M[S]$ such that:

(i) for all $\beta \in \mathrm{dom}(f)$ and all $1 \leq i \leq r$, $|f(\beta) \cap B_i| = 1$

(ii) for $1 \leq i \leq r$ the map $f_i : \mathrm{dom}(f) \to M[S]$ defined by $f_i(\beta) := f(\beta) \cap B_i$ is a C-isomorphism.

b) A partial definable function $f : M \to T(M)^{[r]}$ is a *multi-isomorphism* if there are $s \leq r$, a strong multi-isomorphism $\hat{f} : M \to T(M)^{[s]}$ with $\mathrm{dom}(\hat{f}) = \mathrm{dom}(f)$ and a definable antichain A of $T(M)$ such that

$$f(\alpha) = \{x \in A : \exists b \in \hat{f}(\alpha)(b \leq x)\}.$$

c) A partial definable function $f : M \to Z^{[r]}$ where Z is either $\mathcal{C}$ or $\mathcal{L}_n$ for $n < \omega$ is a multi-isomorphism if:

(i) For all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a cell of type $Z^{[r]}$

(ii) The induced function $f_{\min} : M \to T(M)^{[k]}$ (for $k \leq r$) defined by $f_{\min}(\alpha) = \{\min(F) : F \in f(\alpha)\}$ is a multi-isomorphism.

d) A partial definable function $f : M \to \mathcal{I}^{[r]}$ is a multi-isomorphism if:

(i) for all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a cell of type $\mathcal{I}^{[r]}$

(ii) the induced functions $f_l : M \to T(M)^{[s_1]}$ and $f_r : M \to T(M)^{[s_2]}$ sending α respectively to left and right end-points of elements in $f(\alpha)$ satisfy that f_r is a multi-isomorphisms and f_l is either constant or a multi-isomorphism.

e) For $f : M \to Z^{[r]}$ where Z is any of $T(M)$, $\mathcal{C}$ or $\mathcal{L}_n$ for $n < \omega$, we say that f is a *local multi-isomorphism* (*local strong multi-isomorphism*) if for all $\alpha \in \mathrm{dom}(f)$ there is a cone D with $\alpha \in D \subseteq dom(f)$ such that $f \upharpoonright D$ is a multi-isomorphism (resp. strong multi-isomorphism).

Remark 33. By the definition of $T(M)$, the case $f : M \to M^{[r]}$ is contained in part (a) of the previous definition given that we can see this function as a function of the form $f : M \to T(M)^{[r]}$ where $\mathrm{im}(f)$ is a subset of the set of leaves of $T(M)$ which is identified with M itself. Moreover notice that $M[M] = M$. In addition, for $r = 1$, both notions of multi-isomorphism and strong multi-isomorphism coincide with the notion of C-isomorphism. Condition (i) in parts (c) and (d) are used only to guarantee that the functions $f_{\min}, f_l$ and f_r are well defined (it might happen that there is $\alpha \in \mathrm{dom}(f)$ such that $F_1, F_2 \in f(\alpha)$ satisfy $\min(F_1) = \min(F_2)$, so one has to be careful when choosing the multi-degree of $f_{\min}$).

Definition 34. Let Z be one of $T(M), \mathcal{C}$ or $\mathcal{L}_n$ for $n < \omega$. Let r, m and k be positive integers and $f : M^m \to Z^{[r]}$ be a definable partial function. We say that f is *reducible to the family* $(f_1, \dots, f_k)$ if and only if for each $1 \leq i \leq k$,

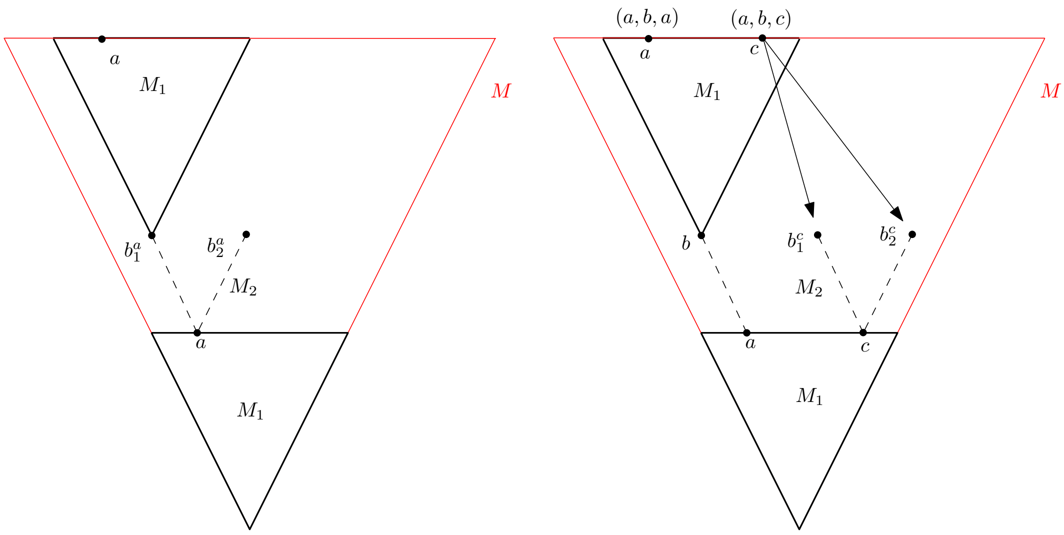
- $f_i : M^m \to Z^{[s_i]}$ is a definable partial function.
- $\mathrm{dom}(f_i) = \mathrm{dom}(f)$.

- $\sum_{i=1}^{k} s_i = r$.
- For every $\alpha \in \text{dom}(f)$, $f(\alpha) = \bigcup_{i=1}^{k} f_i(\alpha)$.

The function f is *irreducible* if there is no non-trivial such reduction of f.

Remark 35. If f is reducible to the family $(f_1, ..., f_k)$ and each f_i is reducible to a family $(g_i^1, ..., g_i^{s_i})$ then f is reducible to the family $(g_1^1, ..., g_1^{s_1}, ..., g_k^1, ..., g_k^{s_k})$. Notice also that if f is reducible to the family $(f_1, ..., f_k)$ and each function f_i in the family is continuous at α then f is continuous at α.

Example 36. Let $A = [0, 1) \cap \mathbb{Q}$ ordered with the usual order. Set T_1 to be the good tree where all branches are order isomorphic to A and $bn(a) = 2$ for all $a \in T_1$ (it is not difficult to show the existence of such object), and T_2 to be the good tree with 3 elements (two incomparable elements above the roor). Let M_1, M_2 be their corresponding induced C-sets. Let M be the C-set $M_1 \rtimes M_2 \rtimes M_1$, i.e., elements in M are triples $(a, b, c) \in M_1 \times M_2 \times M_1$ and where the point a can be identified with its corresponding leaf in the first copy of T_1 and (a, b) with the corresponding leaf in $T_1 \rtimes T_2$. For every $a \in M_1$ we denote by b_1^a and b_2^a the 2 nodes in $T(M)$ which are successors of a (see the figure).



Consider the function $f : M \to T^{[2]}$ mapping (a, b, c) to $\{b_1^c, b_2^c\}$. We show that f is a local multi-isomorphism but not a local strong multi-isomorphism. Notice first that $\text{im}(f)$ is an antichain that corresponds to $M_1 \rtimes M_2$, i.e. to the leaves of $T_1 \rtimes T_2$. Suppose for a contradiction f is a local strong multi-isomorphism. Let $\alpha = (a, b, c) \in M$ and D be a cone containing it. If there where cones B_1 and B_2 satisfying conditions $(i) - (ii)$ in part (a) of definition 32 then $|f(\alpha) \cap B_i| = 1$ for all $i = 1, 2$. By the definition of T_2 this implies that $|B_i| = 1$ which contradicts the fact that $f \restriction D$ is a C-isomorphism since D is a dense cone. To show f is a local multi-isomorphism consider the function $\hat{f} : M \to T(M)$ defined by $\hat{f}(\alpha) = \inf(f(\alpha))$. Let D be the cone $\Gamma_{(a,b)}(\alpha)$ and $A := f(D)$. The set A is a definable antichain since it is the set of all elements in $T(M)$ with a predecessor. It is easy to see that $\hat{f}((a, b, c)) = c$ and since $D \cong M_1 \cong \text{im}(\hat{f})$, we have that $\hat{f}$ is a local strong multi-isomorphism. Moreover $f(\alpha) = \{x \in A : x > b \text{ for some } b \in \hat{f}(\alpha)\}$. It is worthy to notice that f is an irreducible function (this is mainly because the elements $\{b_1^c, b_n^c\}$ have the same imaginary type over $\{c\}$, hence cannot be separated by a definable partial function).

Lemma 37. *Let $f : M \to T(M)^{[r]}$ and $g : M \to T(M)^{[s]}$ be two partial definable functions such that $s < r$, dom$(f) =$ dom(g) and for all $\alpha \in$ dom(f) we have that $f(\alpha)$ is an antichain and for all $b \in f(\alpha)$ there is $a \in g(\alpha)$ such that $a < b$. Then if g is a local multi-isomorphism so is f.*

Proof: Let $\alpha \in \mathrm{dom}(f)$. By definition of local multi-isomorphism, there is a cone D such that $\alpha \in D \subseteq \mathrm{dom}(g)$ and $g \restriction D$ is a multi-isomorphism. This implies that there are a positive integer $s' \leq s$, a strong multi-isomorphism $\hat{g} : D \to T(M)^{[s']}$ and a definable antichain A_0 of $T(M)$ such that $g(\beta) = \{x \in A_0 : \exists b \in \hat{g}(\beta)(b < x)\}$. Consider the following definable subset of $T(M)$

$$A := \{x \in f(D) : \exists b \in g(D)(b \leq x)\}.$$

We claim that $\hat{g}$ and A witness that $f \restriction D$ is a multi-isomorphism. We first show that A is in fact an antichain. For suppose there are $y, x \in A$ such that and $x < y$. Let $\alpha, \beta \in D$ such that $x \in f(\alpha)$ and $y \in f(\beta)$. By assumption there are $a \in g(\alpha)$ and $b \in g(\beta)$ such that $a \leq x$ and $b \leq y$. Since $\hat{g}$ is a strong multi-isomorphism there are cones $B_1, ..., B_{s'}$ in $M[\hat{g}(D)]$ such that the functions $\hat{g}_i : D \to M[\hat{g}(D)]$ defined by $\hat{g}_i(\beta) := \hat{g}(\beta) \cap B_i$ for $1 \leq i \leq s'$ are C-isomorphisms. Moreover, there are $c \in \hat{g}(\alpha)$ and $d \in \hat{g}(\beta)$ such that $c < a \leq x$ and $d < b \leq y$. Since $c, d < y$ they are comparable, but $\hat{g}(D)$ is an antichain, hence $c = d$. Since the cones $B_1, ..., B_{s'}$ are disjoint, there is a unique i in $\{1, ..., s'\}$ such that $c \in B_i$. Hence we have that $\hat{g}_i(\alpha) = c = d = \hat{g}(\beta)$. Since $\hat{g}_i$ is injective we must have that $\alpha = \beta$. But then we have a contradiction since by assumption $f(\alpha)$ is an antichain and $x, y \in f(\alpha)$ are comparable. Now that we know that A is an antichain we simply check that

$$f(\alpha) = \{x \in A : \exists b \in g(\alpha)(b < x)\} = \{x \in A : \exists c \in \hat{g}(\alpha)(c \leq x)\}$$

which shows that f is a local multi-isomorphism. $\square$

Lemma 38. *Let $\Gamma \subseteq M$ be a cone and let $f : \Gamma \to M^{[s]}$ be a definable function such that for some $a \in T(M)$*

$$\{\inf(\gamma, \gamma') : \exists \alpha, \beta \in \Gamma(\gamma \in f(\alpha) \wedge \gamma' \in f(\beta))\} = \{a\}.$$

Then $\Gamma \setminus X_f$ is finite where

$$X_f := \{\alpha \in \Gamma : \exists D_\alpha, B_1, ..., B_s \in \mathcal{C}(\alpha \in D_\alpha \subseteq \Gamma \wedge \forall \beta \in D_\alpha \bigwedge_{i=1}^{s} (|f(D_\alpha) \cap B_i| = 1))\}.$$

Proof: We prove this by induction on s. For $s = 1$, by proposition 29 we have that f is either locally constant or a local C-isomorphism on a cofinite subset X_0 of Γ. This implies that $X_0 \subseteq X_f$, thus $\Gamma \setminus X_f$ is finite. Suppose the lemma is true for all $k < s$ and all definable functions with multi-degree k. Suppose towards a contradiction that $\Gamma \setminus X_f$ is infinite and let D be an infinite cone contained in $\Gamma \setminus X_f$. We spli in two cases.

Suppose for all $\alpha \in D$, there is no $\gamma \in f(\alpha)$ such that $\Gamma_a(\gamma) \cap f(D)$ is infinite. By C-minimality this implies there is a cone $D_0 \subseteq D$ where f is injective. Let $D_1, D_2 \subseteq D_0$ be two disjoint cones. By injectivity we have that such that $f(D_1) \cap f(D_2) = \emptyset$. But by assumption each $f(D_i)$ must be infinite which contradicts the strong minimality at a. Suppose then there are α and $\gamma \in f(\alpha)$ such that $\Gamma_a(\gamma) \cap f(D)$ is infinite. Then $f^{-1}(\Gamma_a(\gamma))$ is infinite too, so let $D_0 \subseteq D$ be a cone contained in $f^{-1}(\Gamma_a(\gamma))$. Define $\hat{f} : U \to M^{[s-1]}$

by $\hat{f}(\alpha) = f(\alpha) \cap \Lambda_a(\gamma)$, which is well defined by the choice of D_0. By induction, there are finitely many elements in $D_0 \setminus X_{\hat{f}}$. For $\alpha \in D_0 \setminus X_{\hat{f}}$ let D_α and $B_1, ..., B_{s-1}$ such that $\bigwedge_{i=1}^{s-1}(|\{x \in \hat{f}(\beta) : x > a\} \cap B_i| = 1)$. Setting $B_s = \Gamma_a(\gamma)$, since $D_\alpha \subseteq D_0$, we have that $\bigwedge_{i=1}^{s}(|\{x \in f(\beta) : x > a\} \cap B_i| = 1)$, which shows $\alpha \in X_f$, a contradiction. □

Lemma 39. *Let $f : M \to Z^{[r]}$ be a definable partial function where Z is either M, T or $\mathcal{L}_0$. If f is a local multi-isomorphism, then f is continuous on a cofinite subset of its domain.*

Proof: The set $A := \{\alpha \in \text{dom}(f) : f \text{ is not continuous at } \alpha\}$ is definable in all cases given that all topologies have a uniformly definable subbasis. Suppose towards a contradiction that A is infinite. Then it contains a cone E. We split in cases depending on the value of Z.

Case $Z = M$**:** Since f is a local multi-isomorphism, without loss of generality we may assume that f is a multi-isomorphism on E. This implies that for $1 \leq ß \leq r$ there are cones B_i such that $|f(\beta) \cap B_i| = 1$ for all $\beta \in E$ and the function $f_i : E \to B_i$ defined by $f(\beta) \cap B_i$ is a C-isomorphism. Given that the cones $B_1, \ldots, B_r$ are pairwise disjoint, $f(E) \subseteq B_1 \times \cdots \times B_r$ is open, which contradicts that $E \subseteq A$.

Case $Z = T$ **or** $Z = \mathcal{L}_0$**:** The case $Z = \mathcal{L}_0$ is reduced to the case of $Z = T$ by the definition of its topology. Again, since f is a local multi-isomorphism, without loss of generality we may assume that f is a multi-isomorphism on E. This implies there is a strong multi-isomorphism $\hat{f} : M \to T^{[s]}$ and a definable antichain S such that $f(\alpha) = \{x \in S : \exists b \in \hat{f}(\alpha)(b < x)\}$ for all $\alpha \in E$. By definition of strong multi-isomorphism there are cones $B_1, \ldots, B_s$ such that for each $1 \leq i \leq n$, $|\hat{f}(\alpha) \cap B_i| = 1$ for all $\alpha \in E$ and the function $\hat{f}_i : E \to B_i$ defined by $\hat{f}(\alpha) \cap B_i$ is a C-isomorphism. Therefore $\hat{f}_1(E), \ldots, \hat{f}_s(E)$ are cones. For $\alpha \in E$ take $(\Lambda_1, ..., \Lambda_r)$ be such that for each $1 \leq i \leq r$ there is $a \in f(\alpha)$ such that $a < \min(\Lambda_i)$. Consider the open set of $T^{[r]}$ given by $U = \{\{a_1, ..., a_r\} : a_i \in (B_j, \Lambda_i)_T, \min(B_j) < \min(\Lambda_i)\}$. This open set contains α. We show that $f(E) \subseteq U$. Let $\beta \in E$. By assumption $f(\beta) = \{x : \exists b \in \hat{f}(\beta)(b < x)\}$, so given that $\hat{f}(E) \subseteq B_1 \times \cdots \times B_s$, we have that for all $b \in f(\beta)$ there is some $1 \leq j \leq s$ such that $b \in B_j$. Moreover given that S is an antichain, $b \notin \Lambda_i$ for all $1 \leq i \leq r$, therefore for each $b \in f(\beta)$ there is one open set (B_j, Λ_i) containing b. This contradicts that f is not continuous at α. □

We are now ready to prove theorem 27:

Proof of theorem 27: Let $f : M \to Z^{[r]}$ be a partial definable function. The proof goes by induction on r and the case $r = 1$ corresponds to theorem 29 and lemma 30. It is worthy to notice that if $Y \subseteq \text{dom}(f)$ is a 1-cell such that $f \upharpoonright Y$ is reducible to a family of functions $(f_1, ..., f_k)$, then on Y the result follows by the induction hypothesis and remark 35, since the multi-degree of each f_i is strictly less than r. As usual, we split in cases depending on the value of Z:

Case $Z = M$**:** For each $\alpha \in \text{dom}(f)$, let T_α be $T[f(\alpha)]$ (the closure of $f(\alpha)$ under inf, which is in particular finite) and L_α be the set of leaves of $T_\alpha \setminus f(\alpha)$. Since there are finitely many isomorphism types of trees T_α, say $T_1, ..., T_m$, we have that

$$\text{dom}(f) = \bigcup_{i=1}^{m} X_i \quad \text{where } X_i := \{\alpha \in \text{dom}(f) : T_\alpha \cong T_i\}.$$

Since each X_i is definable, by theorem 21, X_i decomposes into finitely many 1-cells. Therefore, $\mathrm{dom}(f)$ decomposes into finitely many 1-cells such that $T_\alpha \cong T_\beta$ for all α, β belonging to the same 1-cell. Let Y be such a cell and let $\alpha \in Y$. Since $|L_\alpha| = |L_\beta|$ for all $\beta \in Y$, we let $r_0 = |L_\alpha|$. We may assume that for all $\gamma \in f(\alpha)$ there is $a \in L_\alpha$ such that $a < \gamma$ (see figure 1), for otherwise $f \restriction Y$ can be reduced to the family (f_1, f_2) where

$$\begin{aligned} f_1(\alpha) :=& \quad \{x \in f(\alpha) : \exists a \in L_\alpha (a < x)\} \\ f_2(\alpha) :=& \quad f(\alpha) \setminus f_1(\alpha) \end{aligned}$$

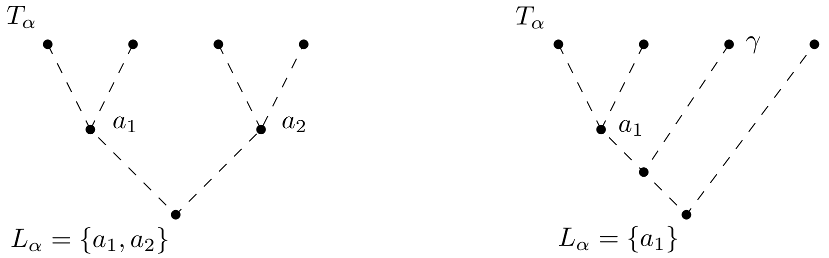


Figure 1: Suppose $f : M \to \binom{M}{4}$. In the left-side example, for every $x \in f(\alpha)$ there is $a \in L_\alpha$ such that $a < \gamma$. In the right-side example there is no $a \in L_\alpha$ such that $a < \gamma$, hence f will be reducible in this case.

and the result follows by induction. Notice that the fact that $T_\alpha \cong T_\beta$ for all $\alpha, \beta \in Y$ implies that f_1, f_2 are well-defined functions. We may suppose furthermore that for all $\alpha \in Y$ and all $a \in L_\alpha$, the cardinality of the set $\{x \in f(\alpha) : a < x\}$ is the same, since if there were at least two such different cardinalities, say s_1 and s_2, $f \restriction Y$ would again be reducible to (f_1, f_2) where

$$\begin{aligned} f_1(\alpha) :=& \quad \{x \in f(\alpha) : \exists a \in L_\alpha |(a < x)| = s_1\} \\ f_2(\alpha) :=& \quad f(\alpha) \setminus f_1(\alpha) \end{aligned}$$

and the result follows by induction (see figure 2).

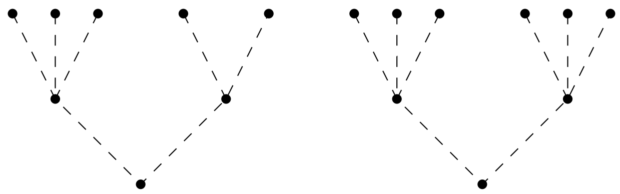


Figure 2: In the left-side example, we have different cardinalities for the elements above $a \in L_\alpha$ so it is reducible. In the right-side example there they are all the same.

Now define the function $g : Y \to T(M)^{[r_0]}$ by $g(\alpha) := L_\alpha$. Since $r_0 < r$, by induction Y decomposes into finitely many 1-cells such that on each infinite cell W, $g \restriction W$ is reducible to a family $(g_1, ..., g_l)$ where each g_i is either locally constant or a local multi-isomorphism. If $l > 1$, then we can reduce $f \restriction W$ to the family $(f_1, ..., f_l)$ where $f_i(\alpha) = \{x \in f(\alpha) : \exists a \in g_i(\alpha)(a < x)\}$, and the result follows by induction. If $l = 1$, the remaining cases correspond to whether $g \restriction W$ is locally constant or a local multi-isomorphism:

• **Case 1)** Suppose that $g \restriction W$ is a local multi-isomorphism. We show this is impossible, i.e., we show that W is empty. Otherwise, let $\alpha_0 \in W$ and D be a cone containing α_0 such that $g \restriction D$ is a multi-isomorphism. Thus, by definition, there are

$k \leq r_0$, a strong multi-isomorphism $\hat{g} : D \to T(M)^{[k]}$ and a definable antichain A such that $g(\beta) = \{x \in A : \exists b \in \hat{g}(\beta)(b \leq x)\}$ for all $\beta \in D$. By definition of strong multi-isomorphism, the set $S = \hat{g}(D)$ is an antichain and there exist cones $B_1, ..., B_k$ in the C-structure $M[S]$ such that for $1 \leq i \leq k$ the function $\hat{g}_i : D \to B_i$ defined by $\hat{g}_i(\beta) = \hat{g}(\beta) \cap B_i$ is a C-isomorphism. In particular, since D is infinite this implies that $\hat{g}(D)$ is infinite.

Claim 40. *For each $b \in S$ the set $\{x \in f(D) : b \leq x\}$ is finite.*

We prove in particular that for $b \in S$ there is $\alpha \in D$ such that $\{x \in f(D) : b \leq x\} = \{x \in f(\alpha) : b \leq x\}$. Since $f(\alpha)$ is finite this shows the claim. For $b \in S = \hat{g}(D)$, let $\alpha \in D$ such that $b \in \hat{g}(\alpha)$. Let B_i be such that $b \in B_i$, so $\hat{g}_i(\alpha) = b$ for some $1 \leq i \leq k$. We show that α satisfies what we want. The right-to-left inclusion is trivial. For the converse, let $x \in f(\beta)$ and $b \leq x$. Then there is $a \in g(\beta)$ such that $a \leq x$ and since $g(\beta) = \{x \in A : \exists c \in \hat{g}(\beta)(b \leq x\}$, there is $c \in S$ such that $c \leq a \leq x$. Since $b, c \leq x$ they are comparable but they both lie in S which is an antichain, which implies $b = c$. Therefore $\hat{g}_i(\alpha) = b = c = \hat{g}_i(\beta)$, and since $\hat{g}_i$ is injective, we have $\beta = \alpha$, which finishes the claim.

But since we have that $\bigcup_{b \in S}\{x \in f(D) : b \leq x\} = f(D)$, $f(D)$ is an infinite definable subset of M which by the density assumption does not contain a cone. This contradicts C-minimality and completes case 1.

- **Case 2)** Suppose that $g \restriction W$ is locally constant. Consider the formula

$$\begin{aligned}\psi(\alpha, a) := \{\alpha \in W : \quad & a \in L_\alpha \wedge \exists D_{\alpha,a}, B_1, ..., B_s \in \mathcal{C}(\alpha \in D_\alpha \subseteq W \ \wedge \\ & \forall \beta \in D_\alpha \textstyle\bigwedge_{i=1}^{s}(|\{x \in f(\beta) : x > a\} \cap B_i| = 1))\}.\end{aligned}$$

Claim 41. *Let $W_1 := \{\alpha \in W : \phi(\alpha, a)$ holds for all $a \in L_\alpha\}$. Then $W \backslash W_1$ is finite.*

Proof: Suppose not and let $D \subseteq W \backslash W_1$ be a cone and $\alpha \in D$. By the choice of W, we may assume that $L_\alpha = L_\beta$ for all $\beta \in D$. Moreover, since L_α is finite, we may also assume that there is $a \in L_\alpha$ such that $\neg\phi(\beta, a)$ for all $\beta \in D$. Then define $f_a : D \to M^{[s]}$ to be the function $f_a(\beta) := f(\beta) \cap \{x \in M : a \in x\}$. The function f_a satisfies all properties of lemma 38, since D is infinite there is $\alpha \in D \cap W_1$ a (notice W_1 corresponds to X_f in lemma 38) which is a contradiction. This shows the claim.

Fix $\alpha \in W$ and take a cone D such that $\alpha \in D \subseteq W$ and $L_\alpha = L_\beta$ for all $\beta \in D$, which exists by the assumption on W. Throwing away finitely many points by the claim, we can assume that $\alpha \notin W_1$. Thus we can define (uniformly) for each $\alpha \in W$, each $1 \leq i \leq s$ and each $a \in L_\alpha$ the function

$$f_\alpha^{a,i} : D_\alpha \to B_i$$

where $D_\alpha \subseteq D$ and B_i are maximal for witnessing $\phi(\alpha, a)$ for all $a \in L_\alpha$. Each $f_\alpha^{a,i}$ is definable so by the induction hypothesis it is either a local C-isomorphism or locally constant on a cofinite subset of D_α. Since L_α is constant, there are finitely many possible combinations for the set $\{f_\alpha^{a,i} : a \in L_\alpha, 1 \leq i \leq s_a\}$ to have l_1 local C-isomorphisms and l_2 locally constant functions where α ranges in W. For those subsets of W where both l_1 and l_2 are different from 0, f is reducible to the family (f_1, f_2), where

$$\begin{aligned}&f_1(\alpha) = \{\gamma \in f(\alpha) : \exists a \in L_\alpha \exists B_0 \in \mathcal{C}(\textstyle\bigvee_{i=1}^{s} \forall \beta \in D_\alpha(f(\beta)^{a,i} = f(\alpha)^{a,i}))\}\\ &f_2(\alpha) = f(\alpha) \setminus f_1(\alpha)\end{aligned}$$

and the result follows by induction. So we may suppose that $f_\alpha^{a,i}$ is either locally constant or a local C-isomorphism for all $a \in L_\alpha$ and all $1 \leq i \leq s_a$. Then, by definition, f is locally constant in the first case and a local multi-isomorphism in the second. This completes this case. Again, notice that if f is locally constant then it is continuous. For local multi-isomorphisms continuity follows by lemma 39. This completes the proof for the case $Z = M$.

Cases $Z = T$ and $Z = \mathcal{L}_0$: Since functions to T or $\mathcal{L}_0$ are in definable bijection, we only consider the case $Z = T$. For $\alpha \in \mathrm{dom}(f)$ set now $T_\alpha := f(\alpha)$. Analogously as in the previous case we decompose $\mathrm{dom}(f)$ into finitely many 1-cells such that $T_\alpha \cong T_\beta$ for all α, β belonging to the same 1-cell. If for such a 1-cell Y and $\alpha \in Y$ the isomorphism type of T_α is not an antichain, then f is reducible to a family of functions $(f_1, ..., f_s)$ where $s-1$ is the height of T_α and each f_i is defined as $f_i(\alpha) = \{a \in T_\alpha : a \text{ has height } i-1\}$, and the result follows by induction. It remains the case where Y is a cell such that T_α is an antichain for all $\alpha \in Y$. We define L_α, g, r_0 and s exactly as in the previous case and by the same argument we are reduced to the cases where there is a 1-cell W such that $g \restriction W$ is either locally constant or a local multi-isomorphism.

• **Case 1)** Suppose that $g \restriction W$ is a local multi-isomorphism. We cannot apply the same argument because the C-structure we have now in the codomain of f might not be dense. Nevertheless this is exactly why the notion of multi-isomorphism is different for M and for $T(M)$. In fact, in this case f is also a local multi-isomorphism by lemma 37.

• **Case 2)** The argument here is the same as in case 2 for $Z = M$. As before, continuity follows by lemma 39.

Cases $Z = \mathcal{C}$ and $Z = \mathcal{L}_n$ for $0 < n < \omega$: Consider the function $\hat{f} : M \to T(M)^{[k]}$ defined by $\hat{f}(\alpha) := \{\min(D) : D \in f(\alpha)\}$. By reducing to a family of functions we may assume that $\hat{f}$ is well-defined for all $\alpha \in \mathrm{dom}(f)$. By the case $Z = T$ and possibly reducing to a family of functions, we may assume that $\mathrm{dom}(\hat{f}) = \mathrm{dom}(f)$ can be finitely decomposed into cells such that on each infinite cell $\hat{f}$ is a local multi-isomorphism or locally constant. Let be X such a cell and $\alpha \in X$. If $\hat{f}$ is a local multi-isomorphism on X then so is f by definition. But by lemma 30 there are no C-isomorphisms from M to $\mathcal{C}$ or $\mathcal{L}_n$ for $0 < n < \omega$, so this situation cannot occur. So assume that $\hat{f}$ is locally constant on X. By C-minimality, since locally all images of f are (unions of) cones with the same base we have that $X = A \cup B$ where

$$A = \{\alpha \in X : \exists D_\alpha \in \mathcal{C}(\alpha \in D_\alpha \wedge \forall \beta, \gamma \in D_\alpha f(\beta) = f(\gamma))\}$$
$$B = \{\alpha \in X : \exists D_\alpha \in \mathcal{C}(\alpha \in D_\alpha \wedge \forall \beta \neq \gamma \in D_\alpha f(\beta) \cap f(\gamma) = \emptyset)\}$$

Clearly $f \restriction A$ is locally constant. We claim that B is finite. For if not, let $B' \subseteq B$ be a cone. Since B' is infinite, there are $\alpha \neq \beta \in B'$ and cones B_α, B_β containing α and β respectively such that $\beta \notin B_\alpha$ and $\alpha \notin B_\beta$. By assumption we have that $f(B_\alpha)$ and $f(B_\beta)$ are two disjoint infinite sets of cones at the same node of $T(M)$, which contradicts C-minimality. This implies already the continuity condition.

Case $Z = \mathcal{I}$: By reducibility and cell decomposition we can assume that the functions $f_l : M \to T(M)^{[s_1]}$ and $f_l : M \to T(M)^{[s_2]}$ sending α respectively to the set of left and right end-points of $f(\alpha)$ are well-defined and that $f_l(\alpha)$ and $f_r(\alpha)$ are cells in $M[f_l(\alpha)]$ and $M[f_r(\alpha)]$ respectively. By the case $Z = T$ and possibly reducing to a family of functions, we may assume that $\mathrm{dom}(f_l) = \mathrm{dom}(f_l) = \mathrm{dom}(f)$ can be finitely decomposed into cells such that on each infinite cell f_l and f_r are continuous and either local multi-isomorphisms or locally constant. This implies already the continuity of f by definition of its topology. Let be X such a cell and $\alpha \in X$. If f_r is a local multi-isomorphism and f_l is a either a local multi-isomorphism or locally constant, then f is a local multi-isomorphism by definition. Notice that the combination f_l being a local multi-isomorphism and f_r being locally constant is impossible (it contradicts that T is a tree). Clearly if both functions are locally constant then f is also locally constant. □

3 Dimension and the cell decomposition theorem

We start this section showing that dense C-minimal structures have a well-behaved topological dimension. Lemma 31 and the rough cell decomposition (proposition 25 and corollary 26) proved in section **??** are sufficient to prove this. Through the section M will be a dense C-minimal structure. We start with a definition:

Definition 42. Let X be a subset of M^n. The dimension of X, denoted $\dim(X)$, is the maximal integer $k \leq n$ such that there is a projection $\psi : M^n \to M^k$ (which does not need to be onto the first k coordinates) for which $\pi(X)$ has non-empty interior in M^k.

By density, a definable subset $D \subseteq M$ has dimension 1 if and only if it contains a cone, i.e., if and only if it is infinite. We cannot generalise this to higher dimensions. In fact, the existence of C-minimal structures having *bad functions*, i.e., definable partial functions $f : M \to T$ containing a cone in their domain for which f is a C-isomorphism, shows that we can have n-cell D of type $(Z_1^{[r_1]}, \ldots, Z_n^{[r_n]})$ where $Z_i \neq M$ for all $1 \leq i \leq n$ such that $\dim(D) < n$. Indeed, if f is such a function and D is a cone on which f is a C-isomorphism, the definable set defined by $\{(\alpha, \beta) \in M^2 : \beta \in \Lambda_{f(\alpha)}\}$ is a cell of type $(\mathcal{C}, \mathcal{L}_0)$ with empty interior in M^2. Still, we are able to prove the following theorem.

Theorem 43. *Let $X_1, ..., X_m$ be definable subsets of M^n. Then*

$$\dim\left(\bigcup_{i=1}^{m} X_i\right) = \max\{\dim(X_i) : 1 \leq i \leq m\}.$$

Proof: Let $X = \bigcup_{i=1}^m X_i$ and $k = \dim(X)$. The proof goes by induction on n.

• For $n = 1$, if X is finite there is nothing to prove. If X is infinite, there is some $1 \leq i \leq m$ such that X_i is infinite and hence contains a cone, i.e., it has non-empty interior.

• Assume the result for all integers less than n. Given that $X_i \subseteq X$ for all $1 \leq i \leq m$ we have that $k \geq \max\{\dim(X_i) : 1 \leq i \leq m\}$. For the converse, we must find some $1 \leq i \leq m$ such that $\dim(X_i) = k$. By corollary 26, we may assume that each X_i is an almost n-cell for all $1 \leq i \leq m$. We split in two cases. Suppose first that $k < n$. Then there is a projection $\psi : M^n \to M^k$ such that $\psi(X)$ has non-empty interior. Since $\psi(X) = \bigcup_{i=1}^m \psi(X_i)$, by induction hypothesis there is some $1 \leq i \leq m$ such that $\psi(X_i)$ has non-empty interior. Therefore $\dim(X_i) = k$. Now suppose that $k = n$. We may assume that for all $1 \leq i \leq m$, if $(Z_1^{[r_2]}, \ldots, Z_n^{[r_n]})$ is the type of X_i then $Z_n \neq M$. This is because if $Z_n = M$ then $X \setminus X_i$ will still have non-empty interior.[3] Let $U \subseteq X$ be a basic open and $\pi : M^n \to M^{n-1}$ be the projection onto the first $n-1$ coordinates. For $(\alpha, \beta) \in U \subseteq M^{n-1} \times M$ let $U^\beta := \{(x, \beta) \in U\}$. Since U is a basic open, $\pi(U^\beta)$ is a

3. One can prove this formaly by induction on n. The base case corresponds to the fact that the union of two finite sets is finite. The inductive case reduces one of the coordinates to the base case.

basic open too. Moreover, since $\pi(U^\beta) = \bigcup_{i=1}^m \pi(X_i \cap U^\beta)$, by induction hypothesis there is $1 \leq i \leq m$ such that $\pi(X_i \cap U^\beta)$ has interior. Let $V \subseteq \pi(X_i \cap U^\beta)$ be an open set. Hence by definition of U^β we have that $V \times \{\beta\} \subseteq X_i$. Now since X_i is an almost n-cell, it is of the form $X_i = \{(x, y) \in \pi(X_i) \times M : y \in \bigcup f(x)\}$ where $f : \pi(X_i) \to Z^{[r]}$ is a definable function and Z is either $\mathcal{C}, \mathcal{I}$ and $\mathcal{L}_l$ for $l < \omega$. Consider the definable function $h : V \to Br(\beta) \cup \{-\infty\}$ where $h(\gamma)$ is the base of the maximal cone containing β which is contained in $f(\gamma)$. By lemma 31 there are $V' \subseteq V$ containing α and $b < \beta$ such that $h(\gamma) < b$ for all $\gamma \in V'$. We show that $V' \times \Gamma_b(\beta) \subseteq X_i$, which shows that X_i has non-empty interior. For $(\gamma, \delta) \in V' \times \Gamma_b(\beta)$, we have that $\gamma \in \pi(X_i)$ and that $\beta \in f(\gamma)$. Now $\delta \in \Gamma_b(\beta)$ which by assumption is contained in $f(\gamma)$, hence $\delta \in f(\gamma)$ so $(\gamma, \delta) \in X_i$. □

Definition 44. Let X be a subset of M^n and $\psi : M^n \to M^k$ be a projection. We say ψ is finite for X if for every $\alpha \in X$ the set $\{\beta \in X : \psi(\alpha) = \psi(\beta)\}$ is finite. We say ψ is open for X, if $\psi(X)$ is open in M^k.

It is important to remark here that given a subset $X \subseteq M^n$ there is a unique natural number k for which a projection $\pi : M^n \to M^k$ is finite and open. Indeed, if there is another projection $\rho : M^n \to M^s$ with $k < s$, then one can prove that π is not finite for X using one of the components not dropped by ρ and dropped by π. If $s < k$ the argument is analogous exchanging the role of π and ρ.

Theorem 45. *Let X be a definable subset of M^n. Then X can be partitioned into definable sets $X_1, ..., X_m$ for which there are natural numbers $k_1, \ldots, k_m$ such that for $1 \leq i \leq m$ there is a projection $\pi_i : M^n \to M^{k_i}$ which is finite and open for X_i.*

The proof goes by induction on n.

• For $n = 1$, by 21 there is a 1-cell decomposition $\{X_1, \ldots, X_m\}$ of X. If X_i is a 1-cell of type $M^{[r_i]}$ then set $k_i = 0$. If X_i is a 1-cell of type $Z^{[r_i]}$ for $Z \neq M$, set $k_i = 1$. By density, X_i contains a cone and the identity function is a finite and open projection for X_i.

• Assume the result for all $n' < n$. By proposition 25 X can be decomposed into finitely many almost n-cells, so without loss of generality we may assume that X is an almost n-cell. Let $(Z_1^{[r_1]}, \ldots, Z_n^{[r_n]})$ be the type of X. We split in cases.

Case 1: Suppose first that there is $1 \leq i_0 \leq n$ such that $Z_{i_0} = M$. Let $\pi : M^n \to M^{n-1}$ be the projection dropping the i_0^{th} coordinate. By induction hypothesis we have that $\pi(X)$ is partitioned into definable sets $Y_1, \ldots, Y_m$ such that for each $1 \leq i \leq m$ there are projections $\pi_i : Y_i \to M^{k_i}$ which are open and finite for Y_i. For each $1 \leq i \leq m$ let $X_i := \pi^{-1}(Y_i) \cap X$. Then $X_1, \ldots, X_m$ form a definable partition of X. We claim that $\pi_i \circ \pi : M^n \to M^{k_i}$ are finite and open projections for X_i for all $1 \leq i \leq m$. That they are open follows directly since $\pi_i \circ \pi(X_i) = \pi_i(Y_i)$ which is open by assumption. Take $\alpha \in X_i$. Since π_i is finite for $\pi(X_i)$, the set $\{\beta \in \pi(X_i) : \pi_i(\beta) = \pi_i(\pi(\alpha))\}$ is finite, say $\beta_1, ..., \beta_s$. Since

$$\{\beta \in X_i : \pi_i \circ \pi(\beta) = \pi_i \circ \pi(\alpha) = \pi^{-1}(\beta_1) \cup \cdots \cup \pi^{-1}(\beta_1),$$

given that $\pi^{-1}(\beta_j)$ is finite for each $1 \leq j \leq s$ by assumption, we have that $\pi_i \circ \pi$ is finite for X_i for each $1 \leq i \leq m$.

Case 2: Suppose that $Z_i \neq M$ for all $1 \leq i \leq n$. Let $\pi : M^n \to M^{n-1}$ now denote the projection onto the first $n-1$ coordinates. Consider the following definable subset of X:

$$W = \{(\alpha, \beta) \in \pi(X) \times M : \text{ there is a basic open } B \text{ of } M^{n-1} \text{ s.t.} \alpha \in B \text{ and } B \times \{\beta\} \subseteq X\}.$$

We show that W is open and that $V := X \setminus W$ can be partitioned into finitely many sets satisfying the conditions of the theorem. This completes the proof since the identity function witnesses the result for W. Let $f : \pi(X) \to Z_n^{[r_n]}$ be the definable function such that $X = \{(\alpha, \beta) \in \pi(X) \times M : \beta \in \bigcup f(\alpha)\}$. Take $(\alpha, \beta) \in W$ and B the maximal box such that $\alpha \in B$ and $B \times \{\beta\} \subseteq X$. Consider the function $h : B \to Br(\beta) \cup \{-\infty\}$ sending $\gamma \in B$ to the base of the maximal cone contained in $f(\gamma)$ which contains β. By lemma 31, there are a open set $D \subseteq B$ containing α and $b < \beta$ such that for every $\gamma \in D$ we have that $h(\gamma) < b$. We show that $D \times \Gamma_b(\beta) \subseteq W$. Take $(\gamma, \delta) \in D \times \Gamma_b(\beta)$. By assumption, $(\gamma, \beta) \in W$ which implies that $\Gamma_b(\beta) \subseteq f(\gamma)$ and hence $(\gamma, \delta) \in W$. This shows W is open.

It remains to show the theorem for V. By definition, for every $(\alpha, \beta) \in V$ we have that the set

$$V_\beta := \{x \in \pi(V) : (x, \beta) \in V\}$$

has empty interior. The idea here is to apply the induction hypothesis to $\pi(V)$ and take as projections $\pi \times id$. To do this correctly, we have to force a partition using the lexicographic order on the set of possible projections from with domain M^{n-1}. So for $\theta \in {}^{n-1}2$ we let π_θ be the projection of M^{n-1} onto those coordinates for which $\theta(i) = 1$. Providing ${}^{n-1}2$ with the lexicographic order, for each $\beta \in M$ and each $\theta \in {}^{n-1}2$ we define by induction on ${}^{n-1}2$ the sets

$$S_\theta^\beta := \{\alpha \in V_\beta : |\pi_\theta^{-1}(\pi_\theta(\alpha)) \cap V_\beta| \text{ is finite and } \alpha \notin S_\mu^\beta \text{ for any } \mu < \theta\}.$$

Let $\epsilon \in {}^{n-1}2$ the function for which $\epsilon(i) = 1$ for all $1 \leq i < n$, so that π_ϵ is the identify function. By definition, π_θ is finite for S_θ^β and $S_\theta^\beta \cap S_\mu^\beta = \emptyset$ for $\theta \neq \mu$. Since $V_\beta \subseteq M^{n-1}$, by induction hypothesis, we have that $V_\beta = \bigcup\{S_\theta^\beta : \theta < \epsilon\}$, give that S_ϵ^β has interior and V_β does not. Hence the sets

$$V_\theta := \{(\alpha, \beta) \in V : \alpha \in S_\theta^\beta\}$$

form a definable partition of V for each $\theta < \epsilon$. Let $rk(\theta)$ be $\sum_{i=1}^{n-1} \theta(i)$ and define the projection $\widetilde{\pi}_\theta : M^n \to M^{rk(\theta)+1}$ by $\widetilde{\pi}_\theta(\alpha, \beta) := (\pi_\theta(\alpha), \beta)$. This projection is finite for V_θ given that π_θ is finite for S_θ^β. Fix $\theta < \epsilon$. Since $\theta < 1$, $rk(\theta) < n - 1$, so $rk(\theta) + 1 < n$. Thus $\widetilde{\pi}_\theta(V_\theta) \subseteq M^{rk(\theta)+1}$, so by induction hypothesis $\widetilde{\pi}_\theta(V_\theta)$ is a disjoint union of finitely many definable sets $V_\theta^1, \ldots, V_\theta^{s_\theta}$, each with a finite open projection ψ_j^θ. Then $\psi_j^\theta \circ \widetilde{\pi}_\theta$ is a finite open projection for $\widetilde{\pi}_\theta^{-1}(V_{\theta,j})$. Since V_θ is the disjoint union of the sets $\widetilde{\pi}_\theta^{-1}(V_\theta^j)$, this shows the result for V. □

We prove now the cell decomposition theorem for dense C-minimal structures. The only difference with the rough version we already proved is piecewise continuity of definable functions. We include the proof for completeness which follows the same ideas of [HM94] simplifying some of the arguments.

Theorem 46. *Let X be a definable subset of M^n. Then*

- (A_n) *for $X_1, \ldots, X_m$ definable subsets of X, there is a cell decomposition $\mathcal{D}$ of X respecting $X_1, \ldots, X_m$;*
- (B_n) *for $f : X \to Z^{[r]}$ a definable function where Z is either $M, T, \mathcal{C}, \mathcal{I}$ or $\mathcal{L}_l$ for $l < \omega$, X has a cell decomposition $\mathcal{D}$ such that $f \upharpoonright Y$ is continuous for each $Y \in \mathcal{D}$.*

Proof: We prove (A_n) and (B_n) simultaneously by induction on n. Propositions (A_1) and (B_1) correspond respectively to 21 and 27. Suppose (A_i) and (B_i) for all $i < n$.

(A_n)**:** We start showing the case $m = 0$ or, in other words, that any definable set $X \subseteq M^n$ has a cell decomposition. Let π be the projection onto the first $n-1$ coordinates and $\phi(x, y)$ be the formula defining D where $|x| = 1$ and x corresponds to the variable which is dropped by the projection π. For $\alpha \in \pi(X)$ we let $X_\alpha := \{\beta \in M : M \models \phi(\beta, \alpha)\}$. Then by proposition 21, there is a definable finite partition $\mathcal{P}$ of $\pi(D)$ such that for each $A \in \mathcal{P}$, X_α uniformly decomposes into finitely many 1-cells, say $\psi_1^A(M, \alpha), ..., \psi_{n_A}^A(M, \alpha)$. By (A_{n-1}), we may assume each $A \in \mathcal{P}$ is already an $(n-1)$-cell. Suppose that $\psi_i^A(x, y)$ defines a 1-cell of type $Z^{[r]}$. Then we have a definable function $h_i^A : A \to Z^{[r]}$ defined by $h_i^A(\alpha) = \psi_i^A(M, \alpha)$. By (B_{n-1}), A decomposes into finitely many cells $D_1^A, ..., D_{s_A}^A$ such that $h_i^A \restriction D_j^A$ is continuous for all $1 \leq j \leq s_A$. Doing the same for each ψ_i^A and each $A \in \mathcal{P}$, the set $\mathcal{D} = \{Y_{i,j}^A : A \in \mathcal{P}, 1 \leq i \leq n_A, 1 \leq j \leq s_A\}$ where

$$Y_{i,j}^A = \{(\alpha, \beta) \in D_j^A \times M : \beta \in h_i^A(\alpha)\}$$

is a cell decomposition of X. Now to show (A_n), we may assume without loss of generality that $X_1, \ldots, X_n$ form a definable partition of X. Applying the previous result to each X_i entails (A_n) for X and $X_1, \ldots, X_m$.

(B_n)**:** By theorem 45, we may assume that there is a projection $\psi : M^n \to M^k$ which is finite and open for X. We divide the proof in two parts: part [I] shows the result for $k < n$ by induction on k and part [II] shows the result for $k = n$.

Part [I]: If $k = 0$ the result follows directly from (A_n). Suppose the result for all $0 < i < k < n$. Without loss of generality we can suppose that ψ is a projection onto the first k coordinates (no cells argument will be used here). Consider the definable set

$$\begin{aligned} Z := \{(\alpha, \beta) \in M^k \times M^{n-k} : &\text{ there is an open set } B \text{ containing } \alpha \text{ such that} \\ & B \times \{\beta\} \subseteq X \text{ and } f(x, \beta) \text{ is continuous on } B\}. \end{aligned}$$

By 45, $X \setminus Z$ has a definable partition into sets $X_1, \ldots, X_m$ for which there are projections $\pi_i : M^n \to M^{k_i}$ such that π_i is finite and open for X_i for each $1 \leq i \leq m$. We show that $k_i < k$ for all $1 \leq i \leq m$, so by induction we have the result for $X \setminus Z$. Fix some $1 \leq i \leq m$. Without loss of generality we may assume that π_i is a projection onto the first k_i coordinates where $k_i \leq k$. Suppose towards a contradiction that $k_i = k$. Then, for $(\alpha, \beta) \in X_i$ with $|\alpha| = k$, there is an open set B such that $\alpha \in B$ and $B \times \{\beta\} \subseteq X_i$. Since $(\gamma, \beta) \notin Z$ for all $\gamma \in B$, the function $f(x, \beta) \restriction D$ is not continuous on any open set $D \subseteq B$. By induction, B decomposes into cells $D_1, ..., D_s$ such that $f(x, \beta) \restriction D_i$ is continuous for all $1 \leq i \leq s$. But by theorem 43 there is $1 \leq i \leq s$ such that D_i is open, which contradicts the previous. To complete the induction it remains to show that (B_n) holds for Z. By (A_n), it is enough to show that f is continuous on Z. Let $(\alpha, \beta) \in Z$ be such that $|\alpha| = k$ and $\beta = (\beta_1, \ldots, \beta_{n-k})$. For $1 \leq i \leq n-k$, consider the definable function $h_i : \pi(Z) \to Br(\beta_i) \cup \{-\infty\}$ where $h_i(\gamma)$ is sent to the base of the maximal cone E such that

$$Z \cap \{\gamma\} \times M \times \cdots M \times E \times M \times \cdots M = \{\alpha, \beta\},$$

which exists given that ψ is finite for Z. Let U be an open set of $Z^{[r]}$ containing $f(\alpha, \beta)$. By definition of Z, there is an open B containing α such that $f(B \times \{\beta\}) \subseteq U$. By lemma 31 applied to $h_1 \ldots, h_{n-k}$, there are open sets $B_1, \ldots, B_{n-1}$ contained in B and containing α and $b_i < \beta_i$ such that $h_i(\gamma) < b_i$ for each $1 \leq i \leq n-k$. By assumption we have that

$$Z \cap B \times \Gamma_{b_1}(\beta_1) \times \cdots \Gamma_{b_n}(\beta_n) = B \times \{\beta\},$$

which implies that $f(B \times \Gamma_{b_1}(\beta_1) \times \cdots \Gamma_{b_n}(\beta_n)) \subseteq U$. This completes part [I].

Part [II]: Given that $k = n$, X is open. Let π denote now the projection onto the first $n-1$ coordinates. Consider the definable sets:

$$\begin{aligned} Z_1 := \{(\alpha, \beta) \in M^{n-1} \times M : & \text{ there is an open set } B \text{ containing } \alpha \text{ such that} \\ & B \times \{\beta\} \subseteq X \text{ and } f(x, \beta) \text{ is continuous on } B\} \\ Z_2 := \{(\alpha, \beta) \in M^{n-1} \times M : & \text{ there is a cone } D \text{ containing } \beta \text{ such that} \\ & \{\alpha\} \times D \subseteq X \text{ and } f(\alpha, x) \text{ is continuous on } D \\ & \text{and either a multi-isomorphism or constant}\} \end{aligned}$$

By (A_n), there is a cell decomposition $\mathcal{D}$ of X respecting Z_1 and Z_2. It is enough to show (B_n) for each $Y \in \mathcal{D}$. By part [I], we can assume Y is open. We show first that $Y \subseteq Z_1 \cap Z_2$. Take $(\alpha, \beta) \in Y$ with $|\beta| = 1$. By induction, there is a cell decomposition $\mathcal{D}'$ of $\pi(Y)$ such that the function $f(x, \beta) \restriction Y'$ is continuous for each $Y' \in \mathcal{D}'$. Since Y is open, by theorem 43 there is $Y' \in \mathcal{D}'$ containing an open set U. But then there is $\alpha' \in U$ such that $f(x, \beta)$ is continuous on a neighborhood of α' which shows that $(\alpha', \beta) \in Z_1 \cap Y$. Since $\mathcal{D}$ respects Z_1, this implies that $Y \subseteq Z_1$. Analogously, by induction there is a cell decomposition $\mathcal{D}'$ of $Y_\alpha := \{\delta \in M : (\alpha, \delta) \in Y\}$ such that $f(\alpha, x)$ is continuous on each cell and either a local multi-isomorphisms or locally constant on each infinite cell. Since Y is open, there must be at least one $Y' \in \mathcal{D}'$ such that Y' is infinite. Then for $\beta' \in Y'$ there is a cone D containing β' such that $f(\alpha, x)$ is continuous and either a multi-isomorphism or constant on D. This shows $(\alpha, \beta') \in Y \cap Z_2$, hence $Y \subseteq Z_2$.

Now let $(\alpha, \beta) \in Y$ and U be an open set in $Z^{[r]}$ containing $f(\alpha, \beta)$. By definition of Z_1 there is an open set B containing α such that $f(B \times \{\beta\}) \subseteq U$. Consider the definable function $h : B \to Br(\beta) \cup \{-\infty\}$ where $h(\gamma)$ is the base of the maximal cone D containing β such that $f(\gamma, x)$ is continuous and either a multi-isomorphism or constant on D. Such a cone exists given that $B \times \{\beta\} \subseteq Y$. By lemma 31, there are an open set $B_0 \subseteq B$ containing α and $b < \beta$ such that $h(\gamma) < b$ for all $\gamma \in B_0$. As in previous arguments, it is not difficult to see that $f(B_0 \times \Gamma_b(\beta)) \subseteq U$. □

A Appendix

Proof of lemma 9: Suppose towards a contradiction there is such an α. By C-minimality, S is equal to a disjoint union of Swiss cheeses $\bigcup_{k=1}^{n} H_k$, where each H_k is of the form $G_k \setminus \bigcup_{i=1}^{s_k} D_i^k$, all G_k and D_i^k are either cones or 0-level sets and $D_i^k \cap D_j^k = \emptyset$ for $i \neq j$. By assumption, for each $N < \omega$ there is a sequence of nodes $a_1 < \cdots < a_N$ in α satisfying (1). Fixing one sequence for each N, define functions $f_N : \{a_1, \ldots, a_N\} \to \mathcal{P}(\{H_1, \ldots, H_n\}) \setminus \{\emptyset\}$ as follows:

$$H_k \in f_N(a_i) \text{ if and only if } \Lambda_{a_i}(\alpha) \cap H_k \neq \emptyset.$$

Since $\Lambda_{a_i}(\alpha) \cap S \neq \emptyset$ we have that $f_N(a_i) \neq \emptyset$. We show that for all N and each $J \in \mathcal{P}(\{H_1, \ldots, H_n\}) \setminus \{\emptyset\}$

$$|f_N^{-1}(J)| \leq (n \cdot \max\{s_k + 1 : 1 \leq k \leq n\})^n.$$

This contradicts the fact that N can be arbitrarily big since it bounds the cardinality of $\mathrm{dom}(f_N)$ independently of N. Fix N and $J \in \mathcal{P}(\{H_1, \ldots, H_n\}) \setminus \{\emptyset\}$. Take $H_k \in J$ with k minimal. To each $a_i \in f^{-1}(J)$ we associate a sequence S using the following algorithm:

1. Start with S being the empty sequence and set $x = k$.
2. If $H_x \subseteq \Lambda_{a_i}(\alpha)$ set $S = S^\frown(H_x)$ and stop. Otherwise go to step 3.
3. If there is a minimal r such that $D_r^x \subseteq \Lambda_{a_i}(\alpha)$, set $S = S^\frown(H_x, D_r^x)$ and stop. Otherwise go to step 4.
4. Take the least r such that $\Lambda_{a_i}(\alpha) \subseteq D_r^x$. Then there is a minimal s such that $H_s \subseteq D_r^x$ and $\Lambda_{a_i}(\alpha) \cap H_s \neq \emptyset$. Set $S = S^\frown(H_x, D_r^x)$ and then $x = s$. Go to step 2.

The algorithm always ends by 1 and the fact that J is finite. Suppose towards a contradiction that there exist a_i and a_j with the same associated sequence. Then there is either H_s or D_r^s at the end of the sequence which is contained in both $\Lambda_{a_i}(\alpha)$ and $\Lambda_{a_j}(\alpha)$. This contradicts that $\Lambda_{a_i}(\alpha) \cap \Lambda_{a_j}(\alpha) = \emptyset$. Therefore, a bound for all possible such sequences gives us also a bound for $|f^{-1}(J)|$. In particular, we have a crude bound like $|f^{-1}(J)| \leq (n \cdot \max\{s_k + 1 : 1 \leq k \leq n\})^n$. $\square$

Proof of lemma 10: Suppose towards a contradiction there are such sequences of arbitrarily large length. Consider the partial type

$$\Sigma(z) = \{\exists x_0 \ldots \exists x_{N-1} \exists y_0 \ldots \exists y_{N-1} \phi_N(x_0, \ldots, x_{N-1}, z, y_0, \ldots, y_{N-1}) : N < \omega\}.$$

By assumption, $\Sigma(z)$ is consistent. Let α be an element realizing Σ in a elementary extension M' of M. Therefore, there are $(\alpha_i \in M' : i < \omega)$ and $(\beta_i \in M' : i < \omega)$ satisfying $\phi_N(\alpha_0, \ldots, \alpha_{N-1}, \alpha, \beta_0, \ldots, \beta_{N-1})$ for all N. This is impossible since for each $i < \omega$ the set $S = \{x \in M' : f(x) \notin \alpha\}$ satisfies

$$f(\beta_i) \in \Lambda_{\inf(\alpha_i,\alpha)}(\alpha) \cap S \qquad \text{and} \qquad f(\alpha_i) \in \Lambda_{\inf(\alpha_i,\alpha)}(\alpha) \cap (M' \setminus S)$$

which contradicts lemma 9. $\square$

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